may be written

$$F^{\alpha\beta}F_{\mu\alpha|\beta} + 2BF^{\alpha\beta}F_{\alpha\beta|\mu} = \tfrac{1}{2}F^{\alpha\beta}F_{\mu\alpha|\beta} + \tfrac{1}{2}F^{\beta\alpha}F_{\mu\beta|\alpha} + 2BF^{\alpha\beta}F_{\alpha\beta|\mu} = \tfrac{1}{2}F^{\alpha\beta}(F_{\mu\alpha|\beta} - F_{\mu\beta|\alpha} + 4BF_{\alpha\beta|\mu}) \tag{10.67}$$

If we choose $B = \frac{1}{4}$, the expression in the parenthesis becomes

$$\tfrac{1}{3}(F_{\mu\alpha|\beta} - F_{\mu\beta|\alpha} + F_{\alpha\beta|\mu}) = \{F_{\mu\alpha|\beta}\} \tag{10.68}$$

which is zero, by virtue of the second set of Maxwell's equations (10.46). Therefore the choice $A = 1$, $B = \frac{1}{4}$ gives an appropriate $S_\mu{}^\nu$:

$$c^2 S_\mu{}^\nu = F_{\mu\alpha}F^{\alpha\nu} + \tfrac{1}{4}g_\mu{}^\nu F^{\alpha\beta}F_{\alpha\beta} \tag{10.69}$$

The complete matrix $T^{\mu\nu} = M^{\mu\nu} + S^{\mu\nu}$ is thus

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu + \frac{1}{c^2}(F^\mu{}_\alpha F^{\alpha\nu} + \tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}) \tag{10.70}$$

There is clearly no problem at all involved in generalizing (10.70) to a true four-dimensional tensor as there was with the previous example of the perfect fluid. As with the perfect fluid, we can interpret c^2S^{00} as the energy density of the electromagnetic field; a brief calculation using the explicit form (4.48) for $F^{\mu\nu}$ yields

$$c^2 S^{00} = \frac{\mathbf{E}^2 + \mathbf{H}^2}{2} \tag{10.71}$$

which agrees with the familiar expression for electromagnetic energy density used in elementary electrodynamics. Similarly, the ith component of the momentum density of the electromagnetic field may be identified with cS^{0i}, which, by virtue of (10.69), is

$$cS^{0i} = \frac{1}{c}(\mathbf{E} \times \mathbf{H})^i \tag{10.72}$$

The vector $\mathbf{E} \times \mathbf{H}$ is the familiar Poynting vector of electrodynamics which represents momentum density, so we again have consistent agreement with classical electrodynamics. In summary, we may write the complete divergenceless energy-momentum tensor of matter field plus electromagnetic field as

$$T^{\mu\nu} = M^{\mu\nu} + \begin{pmatrix} h & \mathbf{g} \\ \mathbf{g} & S \end{pmatrix} \tag{10.73}$$

where c^2h and $c\mathbf{g}$ are the energy and momentum densities of the electromagnetic field. Note the similarity between this and the fluid tensor (10.45).

Finally, let us note that the electromagnetic field represented by $S^{\mu\nu}$ may exist in the absence of any charged material, in which case it is evident that $S^{\mu\nu}$ is divergenceless. (The reader may check that this follows directly from Maxwell's equations in vacuum.) Thus the tensor $T^{\mu\nu} = S^{\mu\nu}$ is appropriate to describe a *free electromagnetic field.*

We have again in the statement that the energy-momentum tensor is divergenceless, $T^{\mu\nu}{}_{|\nu} = 0$, an elegant formulation of the interaction between electromagnetic fields and matter. It may easily be generalized to an arbitrary coordinate system, as before, by writing in covariant tensor form

$$T^{\mu\nu}{}_{\|\nu} = 0 \tag{10.74}$$

Using the preceding three examples in the framework of special theory relativity as a guide, we now make the following assumption: By including all significant physical quantities in the complete energy-momentum tensor, i.e., matter, fluid pressure, electromagnetic fields, etc., we obtain a *zero-divergence tensor* in flat space. If any quantity is omitted, it manifests itself as a force and the energy-momentum tensor cannot be considered complete and does not have a zero divergence. According to this view, physical quantities influence each other by exchanging energy and momentum in such a way as to keep the divergence of $T^{\mu\nu}$ equal to zero; i.e., total energy and momentum are conserved. We conclude that $T^{\mu\nu}$ concisely characterizes the nongravitational energy content of space. As such it is evidently a natural choice for the "tensor representing energy content of space" term in the symbolic gravitational equation (10.6). The choice of this tensor is further motivated by noting: (1) The T^{00} component of the matter tensor is ρ, the analogue of the right side of Poisson's equation (10.3). (2) The three examples we considered, (10.7), (10.41), and (10.70), involve tensors which are symmetric and, of course, second-rank, like the contracted Riemann tensor which appears in the free-space field equations. Thus we assume that the gravitational field equations have the form

$$\text{Properties of space geometry} = T^{\mu\nu} \tag{10.75}$$

where $T^{\mu\nu}$ is divergenceless and encompasses all physical quantities, except gravity, that contribute to the energy content of space. The physical interpretation of $T^{\mu\nu}{}_{\|\nu} = 0$ in a *curved* Riemann space will be taken up in the next chapter.

10.4 The Field Equations in Nonempty Space

In the previous section we motivated a choice for the term of the symbolic gravitational equation (10.6), which describes the energy content of space. In this section we must obtain a suitable tensor to describe the geometry of space in the presence of an energy field.

The simplest choice for the geometric term might seem to be the contracted Riemann tensor, which we used in the free-space field equations of Chap. 5. The field equations would then be

$$R_{\eta\gamma} = (\text{const})T_{\eta\gamma} \tag{10.76}$$

There is, however, a fatal objection to these equations, for $T_{\eta\gamma}$ has zero divergence (if all physically significant quantities are taken into account), while $R_{\eta\gamma}$ does not, as we found in (5.125). Recall, however, that in Sec. 5.8 we also expressed the free-space field equations in terms of the *divergenceless* Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R^{\alpha}{}_{\alpha} = 0 \tag{10.77}$$

It thus appears that the Einstein tensor is appropriate for use in the field equations. In fact, it can be shown that the most general second-rank tensor $B^{\alpha\gamma}$, which has zero divergence, and is constructed entirely from the metric tensor and its first and second derivatives, and is linear in the second derivatives, is a linear combination of the Einstein tensor and the metric tensor:

$$B^{\alpha\gamma} = G^{\alpha\gamma} + \Lambda g^{\alpha\gamma} \tag{10.78}$$

where Λ is an arbitrary constant. For a proof of this see Cartan (Cartan, 1922). We thus take as gravitational equations for nonempty space

$$G^{\alpha\gamma} + \Lambda g^{\alpha\gamma} = (\text{const})T^{\alpha\gamma} \tag{10.79}$$

These equations were first proposed in this form by Einstein in 1917 (Einstein, 1917).

We noted earlier that several properties are demanded of the gravitational equations. One of the requirements is that they reduce to the free-space field equations when $T^{\alpha\gamma}$, the density of energy in space, is zero. This clearly requires that the constant Λ which appears in (10.79) be zero. Actually, this requirement may be relaxed somewhat without a contradiction with experience; in Chap. 11 we shall consider the consequences

of allowing Λ to be a small nonzero constant. But for now we shall assume that Λ is zero. Thus we postulate the field equations

$$G^{\alpha\gamma} = (R^{\alpha\gamma} - \tfrac{1}{2}g^{\alpha\gamma}R) = CT^{\alpha\gamma} \qquad C = \text{const} \tag{10.80}$$

By a small amount of manipulation these equations can be put into an alternative form. Contracting indices in (10.80), we have

$$R^{\alpha}{}_{\alpha} - \tfrac{1}{2}g^{\alpha}{}_{\alpha}R = CT^{\alpha}{}_{\alpha} \tag{10.81}$$

Thus

$$R = -CT^{\alpha}{}_{\alpha} = -CT \tag{10.82}$$

where T is the scalar $T^{\alpha}{}_{\alpha}$, which we shall refer to as the Laue scalar. Using this result we can write the field equations as

$$R^{\alpha\gamma} = C(T^{\alpha\gamma} - \tfrac{1}{2}g^{\alpha\gamma}T) \tag{10.83}$$

Note the symmetry between $T^{\alpha\gamma}$ and $R^{\alpha\gamma}$ evident in (10.80) and (10.83).

In the next section we shall investigate the classical limit of these equations for the case of weak fields and small velocities.

10.5 Classical Limit of the Gravitational Equations

We wish to show in this section that the field equations (10.80) are, as we desired, a generalization of Poisson's classical field equation (10.3). Besides being a validity check on the field equations, the reduction to the classical limit will give us as a by-product the value of the constant C.

Let us consider a field of matter with low proper density, moving at low velocity. The energy-momentum tensor for this situation is obtainable from the special relativistic matter tensor (10.15) if we neglect terms of order $(v/c)^2$ and $\rho_0(v/c)$:

$$T^{\mu\nu} = \begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.84}$$

We shall assume the flow to be stationary and therefore expect the metric to be time-independent. Using the coordinates of special relativity ct, x, y, and z, we consider a time-independent metric which is the sum of the Lorentz metric and a small time-independent perturbation $\epsilon\gamma_{\mu\nu}$,

(10.85) $$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\gamma_{\mu\nu}$$

If we neglect terms of order $\epsilon\rho_0$, the Laue scalar $T^\mu{}_\mu$ is

(10.86) $$T^\mu{}_\mu = \mathrm{Tr}\begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \rho_0$$

and the right side of the field equations is to first order in all the small quantities ρ_0, v/c, $\epsilon\gamma_{\mu\nu}$:

(10.87) $$C(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T) = C\left\{\begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & -\rho_0 & 0 & 0 \\ 0 & 0 & -\rho_0 & 0 \\ 0 & 0 & 0 & -\rho_0 \end{pmatrix}\right\} = \frac{C\rho_0}{2}\,\delta_{\mu\nu}$$

In Sec. 9.1 we found that neglecting second-order terms in $\epsilon\gamma_{\mu\nu}$ gives the following approximate form for the contracted Riemann tensor:

(10.88) $$R_{\mu\nu} \cong \tfrac{1}{2}[\ln(-g)]_{|\mu|\nu} - \begin{Bmatrix} \beta \\ \mu\ \nu \end{Bmatrix}_{|\beta}$$

Thus the approximate field equations may be expressed as

(10.89) $$\tfrac{1}{2}[\ln(-g)]_{|\mu|\nu} - \begin{Bmatrix} \beta \\ \mu\ \nu \end{Bmatrix}_{|\beta} = \frac{C\rho_0}{2}\,\delta_{\mu\nu}$$

Consider first the case $\mu = \nu = 0$. Since we are considering a time-independent metric, the first term of (10.89) is zero, so we are left with the equation

(10.90) $$\begin{Bmatrix} \beta \\ 0\ 0 \end{Bmatrix}_{|\beta} = (g^{\alpha\beta}[00,\alpha])_{|\beta} = -C\frac{\rho_0}{2}$$

The Christoffel symbol of the first kind is defined by

(10.91) $$[00,\alpha] = \tfrac{1}{2}(g_{0\alpha|0} + g_{\alpha0|0} - g_{00|\alpha})$$

Since the Lorentz metric is constant in space and time, this simplifies to

(10.92) $$[00,\alpha] = -\frac{\epsilon}{2}\gamma_{00|\alpha}$$

Furthermore, $\gamma_{\mu\nu}$ is time-independent, so [00,0] is zero. Neglecting second-order terms in $\epsilon\gamma_{\mu\nu}$, we then have

(10.93) $$g^{\beta\alpha}[00,\alpha] = \frac{\epsilon}{2}\gamma_{00|\beta}$$

which is zero for $\beta = 0$. Substituting this in (10.90), we obtain an approximate field equation for γ_{00}:

(10.94) $$\epsilon\sum_{\beta=0}^{3}\gamma_{00|\beta|\beta} = -C\rho_0$$

or by virtue of time independence,

(10.95) $$\epsilon\sum_{i=1}^{3}\gamma_{00|i|i} = -C\rho_0$$

Equation (10.95) is seen to be precisely Poisson's equation (10.3) if we make the identification

(10.96) $$-\frac{\epsilon\gamma_{00}}{C} = \frac{\varphi}{4\pi\kappa}$$

We therefore have established that the classical theory is the limiting case of the time-independent relativistic theory.

If we combine Eq. (10.96) with the result of Sec. 4.3, which relates the classical potential to the metric perturbation according to

(10.97) $$\varphi = \frac{c^2}{2}\epsilon\gamma_{00}$$

we find that

(10.98) $$C = -\frac{8\pi\kappa}{c^2}$$

Thus, by the postulate that the field equations possess the classical equations as a limit case, we have identified the constant of the field equations.

For completeness the reader may check that the entire system of Eqs.

(10.89) is satisfied by the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1+\epsilon\gamma_{00} & & & \\ & -1+\epsilon\gamma_{00} & & \\ & & -1+\epsilon\gamma_{00} & \\ & & & -1+\epsilon\gamma_{00} \end{pmatrix} \tag{10.99}$$

giving an approximate line element of the form

$$ds^2 = \left(1+\frac{2\varphi}{c^2}\right)c^2\,dt^2 - \left(1-\frac{2\varphi}{c^2}\right)d\sigma^2 \tag{10.100}$$

which is of the same form as the approximate free-space line element (9.44) for a spherically symmetric field.

The most important result of this section is the identification of the constant C. We reiterate the Einstein field equations in their explicit form

$$G^{\alpha\gamma} = R^{\alpha\gamma} - \tfrac{1}{2}g^{\alpha\gamma}R = -\frac{8\pi\kappa}{c^2}T^{\alpha\gamma} \tag{10.101a}$$

or equivalently,

$$R^{\alpha\gamma} = -\frac{8\pi\kappa}{c^2}(T^{\alpha\gamma} - \tfrac{1}{2}g^{\alpha\gamma}T) \tag{10.101b}$$

Exercises

10.1 Show that (10.99) is a solution of the approximate field equations (10.89).

10.2 Discuss the general form that the energy-momentum tensor for a scalar field may take, following the discussion of $T_{\mu\nu}$ for the electromagnetic field.

10.3 Repeat the above exercise for a vector field.

10.4 As in Exercise 4.2, define a complex tensor for the electromagnetic field by

$$\omega_{\mu\nu} = F_{\mu\nu} + i(^*F_{\mu\nu})$$

and denote its complex conjugate by $\bar{\omega}_{\mu\nu}$. Show that $\omega_{\mu\nu}$ is related to

the energy-momentum tensor of the electromagnetic field by

$$-\frac{1}{2c^2}\,\mathrm{Re}\,(\bar{\omega}^{\alpha}{}_{\gamma}\omega_{\alpha\beta}) = S_{\gamma\beta}$$

10.5 Recall from Exercise 4.3 that $\omega_{\alpha\beta}\bar{\omega}^{\alpha\beta} = 0$ and show from this that $S_{\gamma\beta}$ is traceless.

10.6 Perform a "duality rotation" on $F_{\mu\nu}$, defined as

$$F'_{\mu\nu} = F_{\mu\nu}\cos\theta + i(^*F_{\mu\nu})\sin\theta$$

Show that $S_{\mu\nu}$ is unchanged by this transformation. What is the explicit effect on a simple plane wave? If $F_{\mu\nu}$ obeys Maxwell's equations, does $F'_{\mu\nu}$? What happens to the invariants discussed in Exercise 4.3?

10.7 The Petrov classification discussed in Chap. 5 relies on the vacuum field equation $R_{\mu\nu} = 0$. In order to extend the classification scheme to nonempty space we introduce the Weyl tensor, $C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha\gamma}R_{\delta\beta} - g_{\alpha\delta}R_{\gamma\beta} + R_{\alpha\gamma}g_{\delta\beta} - R_{\alpha\delta}g_{\gamma\beta} - \frac{1}{3}(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta})R$ (see Prob. 5.2 for some properties of this tensor). Show that it has the same algebraic symmetry properties as the Riemann tensor and automatically satisfies $C^{\alpha}{}_{\mu\alpha\nu} = 0$, the analogue of the vacuum field equations.

10.8 Show from the above that the Petrov classification of vacuum space-times carries over directly to nonempty space-times if the Riemann tensor is replaced by the Weyl tensor.

10.9 Show that Λ introduced in (10.79) must indeed be a constant and cannot be a function of the coordinates.

Problems

10.1 In the canonical theory of fields a Lagrangian density L is obtained which leads to the desired field equations via the Euler-Lagrange method. An explicit form for a canonical energy-momentum tensor can be obtained from L. Use the canonical formalism to obtain $T_{\mu\nu}$ for a scalar field, a general vector field, and the electromagnetic field. What of the gravitational field? Is the canonical $T_{\mu\nu}$ necessarily a symmetric tensor? What physical demand indicates that it should be symmetric? (See Bjorken and Drell, 1965.)

10.2 Consider a spherical body of constant density and total geometric mass m rotating slowly with angular frequency ω. The approximate metric can be obtained from the field equations by working to first order in m and ω. The appropriate general metric form can be inferred by means analogous to those mentioned in the text, Sec. 6.1. The result is

$$ds^2 = A^2(r)c^2\,dt^2 - B^2(r)[dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)] + 2\Omega(r)r^2\sin^2\theta\,d\varphi\,dt$$

Solve the field equations for the functions A, B, and Ω, using appropriate boundary conditions at the surface of the body. Thereby obtain the Lense-Thirring result used in Sec. 7.7 (see Lense and Thirring, 1918; Adams et al., 1974).

Bibliography

Adams, R., J. M. Cohen, R. J. Adler, and C. Sheffield (1973): Analytic Neutron Star Models, *Phys. Rev.*, **D8**:1651.

Adams, R., J. M. Cohen, R. J. Adler, and C. Sheffield (1974): Analytic Pulsar Models, *Astrophys. J.*,

Bauer, H. (1918): Kugelsymmetrische Lösungssysteme der Einsteinschen Feldgleichungen der Gravitation für eine ruhende, gravitierende Flüssigkeit mit linearer Zustandsgleichung, *Sitzber. Akad. Wiss. Wien*, Abt. IIa, **127**:2141–2227.

Bjorken, J. D., and S. D. Drell (1965): "Relativistic Quantum Fields," New York.

Cartan, E. (1922): Sur les Équations de la gravitation d'Einstein, *J. Math. Pures Appl.*, **1**:141–203.

Cohen, J. M. (1968): Angular Momentum and the Kerr Metric, *J. Math. Phys.*, **9**:905.

Einstein, A. (1915): Die Feldgleichungen der Gravitation, *Sitzber. Preuss. Akad. Wiss. Berlin*, pp. 844–847.

Einstein, A. (1917): Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie, *Sitzber. Preuss. Akad. Wiss. Berlin*, pp. 142–152.

Lense, J., and H. Thirring (1918): Über den Einfluss der Eigenrotation der Zentralkörper auf de Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie, *Phys. Z.*, **19**:156.

Weyl, H. (1919): Über die statischen kugelsymmetrischen Lösungen von Einsteins "kosmologischen" Gravitationsgleichungen, *Phys. Z.*, **20**:31–34.

CHAPTER 11

Further Consequences of the Field Equations

This chapter will be a continuation of the investigation of the field equations which we began in Chap. 10. We shall deal with more formal and less completely solved questions than the simple physical problems of the last chapter. The two topics to be discussed are the relation of the equations of motion to the field equations and the nature of general relativistic conservation laws. Neither subject can be considered complete or closed at present, so this chapter serves only as a short introduction to the current work on these problems. Finally we shall briefly discuss the modification of relativity theory proposed by Brans and Dicke.

11.1 The Equations of Motion

Up to now the procedure in determining the motion of a test particle in a given physical situation was the following: (1) We described the distribution of matter and fields by means of the energy-momentum tensor. (2) We calculated the metric field from the Einstein field equations by integrating the Ricci tensor. (3) We found the trajectory of the test particle as a geodesic of the Riemannian geometry. In this method, we thus used two different basic laws: (*a*) The Einstein field equations and (*b*) the postulate of geodesic motion. However, these two basic laws cannot be independent of each other. The test particle considered is a part of the total matter which enters into the energy-momentum tensor and has been split off unnaturally in order to be studied with greater convenience. The Einstein field equations lead to certain differential equations for the energy-momentum tensor which determine its behavior in time and space. Hence, in particular, one should expect that the motion of a test particle should be somehow contained in the field equations. In other words, it seems possible that the postulate of geodesic motion could be deduced from the field equations instead of being axiomatically required.

The first attempt to study the motion of particles from the field equations without the postulate of geodesic motion was made by Einstein and Grommer in 1927. They showed that a singularity of the metric field could not be freely prescribed in space-time but had a specific form as a consequence of the field equations (Einstein and Grommer, 1927). Clearly, a singularity of the field might be interpreted as a material point, and the interrelation of the two fundamental laws of general relativity was thus indicated. The reasoning of Einstein and Grommer was further developed by Einstein, Infeld, and their collaborators and the Russian school of Fock during the thirties and found a final exposition in the work of Infeld and Plebanski (1960).

Let us point out that the basic reason for the interrelation of the two basic laws is the nonlinear character of the field equations of general relativity. Thus we cannot simply add the effects of separate bodies and their fields to obtain a resultant field as we do in the classical theory of gravitation. Instead, we must consider the combined field as an inseparable whole. This is due to the fact that the gravitational field contains energy and must therefore serve as part of its own source, as we noted in Chap. 9. In a linear theory we might create additional solutions by adding or integrating solutions with point singularities. The nonlinear theory precludes such construction. The nonexistence of solutions with arbitrary singularities must strongly affect the dynamics of a material point since such a point may be viewed as a singularity of the field.

Indeed, we shall show in this section that the field equations actually specify unique equations of motion for the case of a point particle in a gravitational field and that the ensuing trajectory of that particle is a geodesic of the corresponding metric in agreement with our previous postulate of geodesic motion. Thus this postulate appears as a consequence of the field equations, and not as an independent axiom of the theory.

Our derivation will follow the method of Levi-Civita (1929). The reader who wishes a more detailed treatment is referred to Infeld and Plebanski (1960). We shall consider the streamlines of the particles in a cloud of dust represented by the matter tensor discussed in Sec. 10.1.

From the field equations (10.80) we are assured that the energy-momentum tensor $T_{\mu\nu}$ has zero divergence:

$$T^{\mu\nu}{}_{\|\nu} = 0 \tag{11.1}$$

Let us investigate the implications of this relation for the matter tensor (10.7), which represents a cloud of dust that does not interact with itself except via gravitation. Setting the covariant derivative of the matter tensor equal to zero, we obtain directly

$$(\rho_0 u^\nu)_{\|\nu} u^\mu + \rho_0 (u^\nu u^\mu{}_{\|\nu}) = (\rho_0 u^\nu)_{\|\nu} u^\mu + \rho_0 A^\mu = 0 \tag{11.2}$$

where A^μ is introduced for later convenience. We also know that the four-vector u^μ has unit magnitude

$$u^\alpha u_\alpha = \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} g_{\alpha\beta} = 1 \tag{11.3}$$

With this relation we can obtain a very beautiful result from (11.2). Differentiation of (11.3) with respect to s gives

$$(u^\alpha u_\alpha)_{|\nu} u^\nu = (u^\alpha u_\alpha)_{\|\nu} u^\nu = 2(u^\alpha{}_{\|\nu} u^\nu) u_\alpha = 0 \tag{11.4}$$

We recognize this as the statement that u_α and A^α are orthogonal:

$$A^\alpha u_\alpha = 0 \tag{11.5}$$

If (11.2) is now multiplied by u_μ, we see with the aid of (11.5) that

$$(\rho_0 u^\nu)_{\|\nu} u^\mu u_\mu + \rho_0 A^\mu u_\mu = (\rho_0 u^\nu)_{\|\nu} = 0 \tag{11.6}$$

That is, the quantity $\rho_0 u^\nu$, which we may interpret as the momentum density, is conserved. This is an interesting result in itself, but it implies even more. From (11.2) we now see that the vector A^μ is identically zero:

$$A^\mu = u^\nu u^\mu{}_{\|\nu} = u^\nu \left(u^\mu{}_{|\nu} + \left\{ {\mu \atop \alpha\ \nu} \right\} u^\alpha \right) = 0 \tag{11.7}$$

The vector u^ν represents a streamline of the dust cloud, and if an individual dust particle is assigned coordinates x^ν, we can identify u^ν with dx^ν/ds. Then (11.7) becomes a constraint on the particle's motion.

$$\frac{dx^\nu}{ds}\frac{\partial}{\partial x^\nu}\left(\frac{dx^\mu}{ds}\right) + \left\{ {\mu \atop \alpha\ \nu} \right\} \frac{dx^\nu}{ds}\frac{dx^\alpha}{ds} = \frac{d^2x^\mu}{ds^2} + \left\{ {\mu \atop \alpha\ \nu} \right\} \frac{dx^\nu}{ds}\frac{dx^\alpha}{ds} = 0 \tag{11.8}$$

In fact the constraint is a complete equation of motion for the dust particle—the geodesic equation which we anticipated.

We have thus shown that the field equations imply a unique equation of motion for the elements of a cloud of dust particles moving under the influence of whatever gravitational field is present.

It is evident from the above that a vanishingly small globule of dust described by a field u^α that is nearly constant over the size of the globule will move on the geodesic. Such a globule serves as a good test body for the following two reasons: (1) its energy-momentum tensor contains only one scalar parameter, the density ρ_0, and is therefore as simple as possible; (2) its internal structure is described entirely by a uniform velocity field and in this sense also is as simple as possible. It should not be surprising that bodies with more complicated internal structure do not necessarily move on geodesics, e.g., extended bodies and bodies with internal motion (see Probs. 11.1 and 11.2).

11.2 Conservation Laws in General Relativity: Energy-Momentum of the Gravitational Field

We have derived the Schwarzschild line elements as one exact solution of the field equations of general relativity theory and discussed some physical implications of these results. There are very few more known exact solutions of the field equations with physical significance. In facing more complicated problems in general relativity, one is forced into approximation methods and numerical procedures. This is certainly not a new feature of relativistic mechanics, for even in classical mechanics not too many significant problems can be solved in closed analytical form. In celestial mechanics the Kepler problem plays the role analogous to the Schwarzschild problem in relativistic astronomy. It can be solved exactly, and is, at the same time, the starting point for many perturbation and approximation procedures which are needed to solve more complicated and practical problems of celestial mechanics. However, in classical mechanics we have, besides the few explicitly solved standard problems, a large number of general principles which often allow qualitative conclusions of great importance without necessitating explicit solutions of the differential equations involved. It is our aim to give somewhat similar general results in the case of the general relativity theory.

We wish to discuss first the question of conservation laws in connection with the field equations. Since the energy-momentum tensor is proportional to the Einstein tensor,

$$G_{\mu\nu} = CT_{\mu\nu} \qquad C = -\frac{8\pi\kappa}{c^2} \tag{11.9}$$

and since the Einstein tensor has a vanishing covariant divergence, we have the important general identity

$$T^{\mu\nu}{}_{\|\nu} = 0 \tag{11.10}$$

Let us briefly review the fact that in a flat space the vanishing of the divergence of a tensor leads always to an interpretation in terms of conservation laws. In a flat space we can by definition use a coordinate system in which the Christoffel symbols vanish everywhere. Hence covariant differentiation reduces in such coordinates to ordinary differentiation. Suppose now that the tensor $T^{\mu\nu}$ is different from zero only in a finite region of space at every moment. Then the support of $T^{\mu\nu}$, that is, the part of the space-time manifold where the tensor is different from zero, can be enclosed in a "four-tube" D^4 on the wall of which $T^{\mu\nu} \equiv 0$. Consider the part Δ^4 of D^4, which is cut off by the hypersurfaces $t = t_i$ and $t = t_f$ (see Fig. 11.1). The boundary of Δ^4 consists

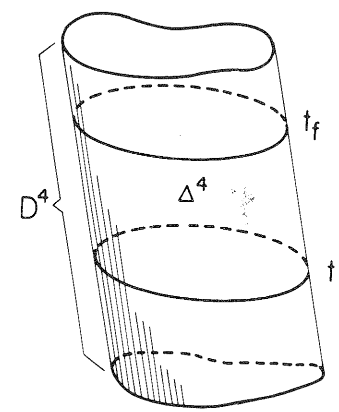

Fig. 11.1

of two three-dimensional spaces at the initial time t_i and the final time t_f and of a part of the wall of D^4 on which the tensor vanishes. Hence, by Gauss's theorem,

$$\int_{\Delta^4} T^{\mu\nu}{}_{|\nu}\, d^4x = \int_{t_f} T^{\mu 0}\, d^3x - \int_{t_i} T^{\mu 0}\, d^3x = 0 \tag{11.11}$$

Thus the vector quantity

$$P^\mu = \int T^{\mu 0}\, d^3x \tag{11.12}$$

does not change with time if we integrate over a three-dimensional space-like region which may change with time, but is chosen in such a way that the tensor $T^{\mu\nu}$ vanishes on its boundary. We may consider P^μ as a conserved quantity and give it a physical interpretation. If $T^{\mu\nu}$ is the energy-momentum tensor, the quantity P^μ can be interpreted as the energy-momentum vector of special relativity and $T^{\mu 0}$ can be interpreted as the density in space of this vector.

Such a straightforward consideration is not possible in a general Riemann space with nonzero Christoffel symbols. Instead, we are forced to search for an alternative conserved quantity to replace P^μ. The reason for this is clear, for in a curved space, energy and field interact, so we can expect only some combination of $T^{\mu\nu}$ and the gravitational field energy to be conserved. We shall find that an expression involving $T^{\mu\nu}$ and a function of the gravitational field variables is indeed conserved. Unfortunately, however, we shall see that there remain problems of interpretation, covariance, and uniqueness connected with the conserved quantity.

To investigate (11.10) in a general Riemann space let us first bring it into explicit form. We have, by definition of covariant differentiation,

(11.13) $$T_\mu{}^\nu{}_{\|\nu} = T_\mu{}^\nu{}_{|\nu} + \begin{Bmatrix} \beta \\ \nu\ \beta \end{Bmatrix} T_\mu{}^\nu - \begin{Bmatrix} \alpha \\ \mu\ \beta \end{Bmatrix} T_\alpha{}^\beta = 0$$

and from Eq. (3.11), we have

(11.14) $$\begin{Bmatrix} \beta \\ \nu\ \beta \end{Bmatrix} = (\log \sqrt{-g})_{|\nu} = \frac{(\sqrt{-g})_{|\nu}}{\sqrt{-g}}$$

Furthermore, the symmetry of $T^{\mu\nu}$ implies that

(11.15) $$\begin{Bmatrix} \alpha \\ \mu\ \beta \end{Bmatrix} T_\alpha{}^\beta = g^{\alpha\tau}[\mu\beta,\tau]T_\alpha{}^\beta = [\mu\beta,\tau]T^{\tau\beta} = \tfrac{1}{2}g_{\tau\beta|\mu}T^{\tau\beta}$$

so we can put (11.13) into simpler form,

(11.16) $$T_\mu{}^\nu{}_{|\nu} + \frac{(\sqrt{-g})_{|\nu}}{\sqrt{-g}} T_\mu{}^\nu - \tfrac{1}{2}g_{\tau\beta|\mu}T^{\tau\beta} = 0$$

that is,

(11.17) $$(\sqrt{-g}\, T_\mu{}^\nu)_{|\nu} - \tfrac{1}{2}g_{\tau\beta|\mu}\sqrt{-g}\, T^{\tau\beta} = 0$$

Because of the field equations (11.10), the last term of (11.17) may be expressed in terms of the Einstein tensor

(11.18) $$\sqrt{-g}\, T_\mu{}^\nu{}_{\|\nu} = (\sqrt{-g}\, T_\mu{}^\nu)_{|\nu} - \tfrac{1}{2}g_{\tau\beta|\mu}\frac{\sqrt{-g}}{C} G^{\tau\beta} = 0$$

(Note that this particularly simple result holds only for the mixed tensor $T_\mu{}^\nu$.) The problem is now clear; if we could write the last term of (11.18) as the ordinary divergence of some quantity $\sqrt{-g}\, t_\mu{}^\nu$, then we should have an identity which would lead to a conservation law via a simple application of Gauss's theorem, as we demonstrated at the beginning of this section.

It should be observed that we have reduced the question of conservation laws in physics to a problem of differential geometry. Indeed, the quantity $\sqrt{-g}\, t_\mu{}^\nu$ which we seek depends only on the metric considered. Once we have determined for each given metric such a quantity, we shall be able to assert conservation laws for a host of physical problems. Indeed, the different physical situations are characterized by the form of the energy-momentum tensor $T^{\mu\nu}$, while the construction which we shall now make will depend only on the structure of the geometric tensor $g^{\mu\nu}$.

The Ricci tensor is built in a very specific nonlinear way from the metric tensor field $g_{\mu\nu}$. In order to exhibit this dependence clearly, it will be useful to vary the field $g_{\mu\nu}$ arbitrarily and to study the effect of this change on the Einstein tensor. We are thus led to methods which are typically used in the calculus of variations. The following considerations will not only lead us to the $\sqrt{-g}\, t_\mu{}^\nu$ terms, but are of independent mathematical interest. They also lay the groundwork for the variational principles in general relativity theory which correspond to the well-known variational formulations in classical dynamics.

We begin our discussion with the scalar density $\mathfrak{R}$ based on the Riemann scalar R:

(11.19) $$\mathfrak{R} = \sqrt{-g}\, R = \sqrt{-g}\, g^{\sigma\rho}R_{\sigma\rho}$$
$$= \sqrt{-g}\, g^{\sigma\rho}\left[\begin{Bmatrix} \alpha \\ \sigma\ \alpha \end{Bmatrix}_{|\rho} - \begin{Bmatrix} \alpha \\ \sigma\ \rho \end{Bmatrix}_{|\alpha} + \begin{Bmatrix} \beta \\ \sigma\ \alpha \end{Bmatrix}\begin{Bmatrix} \alpha \\ \rho\ \beta \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \sigma\ \rho \end{Bmatrix}\begin{Bmatrix} \beta \\ \alpha\ \beta \end{Bmatrix}\right]$$

If D^4 is an arbitrary region in four-space, we can form the invariant integral

(11.20) $$J = \int_{D^4} \mathfrak{R}\, d^4x$$

We can now show that the tensor density $G_{\mu\nu}\sqrt{-g}$ can be obtained as the variational derivative of this expression under a variation of the metric tensor in D^4, which vanishes on the boundary of the region. More precisely, we consider a change of the metric tensor $\delta g_{\mu\nu}$ such that both $\delta g_{\mu\nu}$ and $\delta g_{\mu\nu|\lambda}$ vanish on the boundary of D^4.

We observe that, if we change from the marker system x^α to a marker system $\bar{x}^\alpha$, the metric tensor transforms according to

$$\bar{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu} g_{\alpha\beta} \tag{11.21}$$

The same law is also valid for the varied metric tensor $g_{\mu\nu} + \delta g_{\mu\nu}$. Hence the variation $\delta g_{\mu\nu}$ also transforms like a tensor if we change the coordinate system. The same is true for all tensors built from the metric tensor. In particular, the tensor $R_{\mu\nu}$ will go over into $R_{\mu\nu} + \delta R_{\mu\nu}$ under the variation of the metric tensor, and by the same argument it follows that $\delta R_{\mu\nu}$ will transform as a tensor. Moreover, the variation of the metric tensor field $\delta g_{\mu\nu}$ gives rise to an interesting variation in the Christoffel symbols $\delta \left\{ {\alpha \atop \beta\ \gamma} \right\}$. From the transformation law for $\left\{ {\alpha \atop \beta\ \gamma} \right\}$ as given in (2.5), we recognize that the variation $\delta \left\{ {\alpha \atop \beta\ \gamma} \right\}$ must indeed transform as a tensor,

$$\overline{\delta \left\{ {\alpha \atop \beta\ \gamma} \right\}} = \frac{\partial \bar{x}^\alpha}{\partial x^\kappa}\frac{\partial x^\lambda}{\partial \bar{x}^\beta}\frac{\partial x^\sigma}{\partial \bar{x}^\gamma}\, \delta \left\{ {\kappa \atop \lambda\ \sigma} \right\} \tag{11.22}$$

since the inhomogeneous term in (2.5) depends on the change of the coordinate systems only, but not on the metric used. We already mentioned, in Chap. 2, this important fact that the variations of connections under a change of metric transform like tensors. This will considerably simplify the variational calculations.

Following an idea of Palatini (1919), we now compute the variation of $R_{\mu\nu}$ at a given point by introducing a locally geodesic coordinate system. In such a system all Christoffel symbols vanish at the point considered, by definition. But their variations do not vanish since, with the varied metric, the coordinate system will in general no longer be locally geodesic. Since ordinary and covariant differentiation are the same in a geodesic coordinate system, we have, by definition of $R_{\mu\nu}$,

$$\delta R_{\mu\nu} = \delta \left\{ {\alpha \atop \mu\ \alpha} \right\}_{|\nu} - \delta \left\{ {\alpha \atop \mu\ \nu} \right\}_{|\alpha} = \left(\delta \left\{ {\alpha \atop \mu\ \alpha} \right\}\right)_{\|\nu} - \left(\delta \left\{ {\alpha \atop \mu\ \nu} \right\}\right)_{\|\alpha} \tag{11.23}$$

The terms $\delta R_{\mu\nu}$ and $\left(\delta \left\{ {\alpha \atop \beta\ \gamma} \right\}\right)_{\|\delta}$ are tensors. Hence the extreme terms of (11.23) form a tensor equation. It has been established in a convenient coordinate system, but it must hold in all coordinate systems. Thus

$$\delta R_{\mu\nu} = \left(\delta \left\{ {\alpha \atop \mu\ \alpha} \right\}\right)_{\|\nu} - \left(\delta \left\{ {\alpha \atop \mu\ \nu} \right\}\right)_{\|\alpha} \tag{11.23'}$$

is a generally valid expression for the variation of $R_{\mu\nu}$.

Next let us obtain the variation of $\sqrt{-g}$, which also occurs in the definition of J. The determinant g may be expanded in its elements of the νth column and their cofactors as

$$g = \sum_\mu g_{\mu\nu}\Delta^{\mu\nu} \tag{11.24}$$

where $\Delta^{\mu\nu}$ is the cofactor of $g_{\mu\nu}$ and ν is any fixed column index. Clearly, then,

$$\frac{\partial g}{\partial g_{\mu\nu}} = \Delta^{\mu\nu} \tag{11.25}$$

and hence

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}}\,\delta g_{\mu\nu} = \Delta^{\mu\nu}\,\delta g_{\mu\nu} \tag{11.25'}$$

On the other hand, we can use the definition of the inverse matrix of $g_{\mu\nu}$ to find

$$g^{\mu\nu} = \frac{1}{g}\Delta^{\mu\nu} \tag{11.26}$$

and hence (10.40′) becomes

$$\delta g = g g^{\mu\nu}\,\delta g_{\mu\nu} \tag{11.27}$$

This result can also be formulated in terms of $\delta g^{\mu\nu}$ by noting that $g^{\mu\nu}g_{\mu\nu}$ is the trace of the invariant Kronecker tensor $g^\mu{}_\nu$ and has the value $\delta^\nu{}_\nu = 4$. Hence

$$\delta(g^{\mu\nu}g_{\mu\nu}) = g_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\,\delta g_{\mu\nu} = 0 \tag{11.28}$$

We have remarked that $\delta g_{\mu\nu}$ is a tensor, and hence $g^{\mu\nu}\,\delta g_{\mu\nu}$ is a scalar. But observe that the variational operator δ and the operation of raising and lowering indices do not commute. Thus the contravariant form of $\delta g_{\mu\nu}$ is not $\delta g^{\mu\nu}$, as is evident from (11.28). We see, however, that

$$g^{\mu\nu}\,\delta g_{\mu\nu} = -g_{\mu\nu}\,\delta g^{\mu\nu}$$

and (11.27) becomes

$$\delta g = -g g_{\mu\nu}\,\delta g^{\mu\nu} \tag{11.29}$$

Thus, finally,

$$\delta \sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\,\delta g = -\tfrac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} \tag{11.30}$$

This completes the computation of the variation of the various factors in J.

The total variation of J can now be written as

$$\begin{aligned} \delta J &= \int_{D^4} \delta(\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu})\,d^4x \\ &= \int_{D^4} [\sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu} + g^{\mu\nu}R_{\mu\nu}\,\delta\sqrt{-g} + R_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}]\,d^4x \end{aligned} \tag{11.31}$$

Using the explicit forms for $\delta R_{\mu\nu}$ and $\delta\sqrt{-g}$ from (11.23′) and (11.30), we obtain

$$\begin{aligned} \delta J &= \int_{D^4} \sqrt{-g}\,g^{\mu\nu}\left[\left(\delta \begin{Bmatrix} \alpha \\ \mu\;\alpha \end{Bmatrix}\right)_{\|\nu} - \left(\delta \begin{Bmatrix} \alpha \\ \mu\;\nu \end{Bmatrix}\right)_{\|\alpha}\right] d^4x \\ &\qquad + \int_{D^4} (R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R)\sqrt{-g}\,\delta g^{\mu\nu}\,d^4x \\ &= \int_{D^4} \sqrt{-g}\,g^{\mu\nu}\left[\left(\delta \begin{Bmatrix} \alpha \\ \mu\;\alpha \end{Bmatrix}\right)_{\|\nu} - \left(\delta \begin{Bmatrix} \alpha \\ \mu\;\nu \end{Bmatrix}\right)_{\|\alpha}\right] d^4x \\ &\qquad + \int_{D^4} G_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}\,d^4x \end{aligned} \tag{11.32}$$

Since the covariant derivative of the metric tensor vanishes, we may rewrite the integrand of the first integral in (11.32) in the form

$$\begin{aligned} &\left[\left(g^{\mu\nu}\,\delta \begin{Bmatrix} \alpha \\ \mu\;\alpha \end{Bmatrix}\right)_{\|\nu} - \left(g^{\mu\nu}\,\delta \begin{Bmatrix} \alpha \\ \mu\;\nu \end{Bmatrix}\right)_{\|\alpha}\right]\sqrt{-g} \\ &\qquad = (v^{\nu}{}_{\|\nu} - w^{\alpha}{}_{\|\alpha})\sqrt{-g} \end{aligned} \tag{11.33}$$

where v^{ν} and w^{α} are contravariant vectors. We use now the general formula (3.12), which allows us to express the covariant divergence of a contravariant vector in the form

$$\sqrt{-g}\,v^{\nu}{}_{\|\nu} = (\sqrt{-g}\,v^{\nu})_{|\nu} \tag{11.34}$$

in which the right side is an ordinary divergence term. Thus the entire integrand of the first integral in (11.32) is seen to be an ordinary divergence, and by partial integration we can express this integral as a surface integral over the boundary of D^4. Since $\delta g^{\mu\nu}$ and $\delta \begin{Bmatrix} \alpha \\ \beta\;\gamma \end{Bmatrix}$ vanish on this boundary, the first integral must therefore be zero. Thus the expression (11.32) for the variation of J reduces to the simple fundamental formula

$$\delta J = \int_{D^4} G_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu}\,d^4x \tag{11.35}$$

We have thus proved that *the Einstein tensor density $G_{\mu\nu}\sqrt{-g}$ is the variational derivative of the invariant J integral over the Riemann scalar density $R\sqrt{-g}$.* We state again the fact that (11.35) is valid only for variations of the metric tensor and its first derivatives, which vanish on the boundary of D^4. For more general variations we should have to add certain surface integrals on the right-hand side.

The value of the variational formula (11.35) for formal transformations and the derivation of identities for the Einstein tensor is obvious. We have found the variation of the integral J by simple considerations of covariance. On the other hand, the integrand $\mathfrak{R}$ of J is a complicated nonlinear function of the $g_{\mu\nu}$ and their first and second derivatives. Using the standard formalism of the calculus of variations (the usual Euler-Lagrange approach), we can express the variational derivative $G_{\mu\nu}$ in terms of partial derivatives of $\mathfrak{R}$ with respect to its variables. Observe, however, that even the second derivatives of the varied functions $g_{\mu\nu}$ enter in $\mathfrak{R}$; this will make the Euler-Lagrange terms rather complicated. It is now of great convenience that we can split $\mathfrak{R}$ into one term $\mathfrak{A}$, which depends only on the $g_{\mu\nu}$ and their first derivatives, and into a term $\mathfrak{B}$, which is a linear combination of derivatives. We shall see that the contribution of the second term of the integrand $\mathfrak{R}$ of J will be reducible to an integral over the boundary of the integration domain D^4. Thus the variations of J and of the integral over the much simpler term $\mathfrak{A}$ will be identical. We proceed now to the determination of $\mathfrak{A}$ and carry out the above program. We first rewrite the definition (11.19) of the Riemann density by use of the identity

$$\begin{aligned} &\sqrt{-g}\,g^{\sigma\rho}\left[\begin{Bmatrix} \alpha \\ \sigma\;\alpha \end{Bmatrix}_{|\rho} - \begin{Bmatrix} \alpha \\ \sigma\;\rho \end{Bmatrix}_{|\alpha}\right] \\ &\quad = \left(\sqrt{-g}\,g^{\sigma\rho}\begin{Bmatrix} \alpha \\ \sigma\;\alpha \end{Bmatrix}\right)_{|\rho} - \left(\sqrt{-g}\,g^{\sigma\rho}\begin{Bmatrix} \alpha \\ \sigma\;\rho \end{Bmatrix}\right)_{|\alpha} \\ &\qquad + \begin{Bmatrix} \alpha \\ \sigma\;\rho \end{Bmatrix}(\sqrt{-g}\,g^{\sigma\rho})_{|\alpha} - \begin{Bmatrix} \alpha \\ \sigma\;\alpha \end{Bmatrix}(\sqrt{-g}\,g^{\sigma\rho})_{|\rho} \end{aligned} \tag{11.36}$$

If we substitute this relation into (11.19), we have achieved the decomposition of $R\sqrt{-g}$ into a divergence term and a term containing at most first derivatives of the $g_{\mu\nu}$. However, we can recombine terms in a very remarkable way and find an elegant expression for the term $\mathfrak{A}$.

We start with the fact that the covariant derivative of $g^{\sigma\rho}$ vanishes; this implies that

$$(11.37)\qquad g^{\sigma\rho}{}_{\|\alpha} = g^{\sigma\rho}{}_{|\alpha} + \begin{Bmatrix}\sigma\\ \alpha\ \beta\end{Bmatrix} g^{\beta\rho} + \begin{Bmatrix}\rho\\ \alpha\ \beta\end{Bmatrix} g^{\sigma\beta} = 0$$

This identity allows us to replace all terms $g^{\sigma\rho}{}_{|\alpha}$ by elements $g^{\mu\nu}$ and Christoffel symbols. Similarly, we may use Eq. (11.14) to express the derivatives of $\sqrt{-g}$ in terms of Christoffel symbols. Thus the last two terms in (11.36) become

$$(11.38)\qquad \begin{Bmatrix}\alpha\\ \sigma\ \rho\end{Bmatrix} (\sqrt{-g}\, g^{\sigma\rho})_{|\alpha} - \begin{Bmatrix}\alpha\\ \sigma\ \alpha\end{Bmatrix} (\sqrt{-g}\, g^{\sigma\rho})_{|\rho}$$

$$= \begin{Bmatrix}\alpha\\ \sigma\ \rho\end{Bmatrix}\left[-\begin{Bmatrix}\sigma\\ \alpha\ \beta\end{Bmatrix} g^{\beta\rho} - \begin{Bmatrix}\rho\\ \alpha\ \beta\end{Bmatrix} g^{\sigma\beta} + \begin{Bmatrix}\beta\\ \alpha\ \beta\end{Bmatrix} g^{\sigma\rho}\right]\sqrt{-g}$$

$$- \begin{Bmatrix}\alpha\\ \sigma\ \alpha\end{Bmatrix}\left[-\begin{Bmatrix}\sigma\\ \rho\ \beta\end{Bmatrix} g^{\beta\rho} - \begin{Bmatrix}\rho\\ \rho\ \beta\end{Bmatrix} g^{\sigma\beta} + \begin{Bmatrix}\beta\\ \rho\ \beta\end{Bmatrix} g^{\sigma\rho}\right]\sqrt{-g}$$

This expression collapses by rearrangement and cancellation to

$$(11.39)\qquad \begin{Bmatrix}\alpha\\ \sigma\ \rho\end{Bmatrix} (\sqrt{-g}\, g^{\sigma\rho})_{|\alpha} - \begin{Bmatrix}\alpha\\ \sigma\ \alpha\end{Bmatrix} (\sqrt{-g}\, g^{\sigma\rho})_{|\rho} = 2\mathfrak{A}$$

with

$$(11.40)\qquad \mathfrak{A} = g^{\rho\sigma}\left[\begin{Bmatrix}\alpha\\ \sigma\ \rho\end{Bmatrix}\begin{Bmatrix}\beta\\ \alpha\ \beta\end{Bmatrix} - \begin{Bmatrix}\alpha\\ \beta\ \rho\end{Bmatrix}\begin{Bmatrix}\beta\\ \alpha\ \sigma\end{Bmatrix}\right]\sqrt{-g}$$

Let us substitute the expression (11.39) into (11.36) and insert (11.36) back into the definition (11.19) of the Riemann scalar density

$$\mathfrak{R} = R\sqrt{-g}$$

We thereby obtain the remarkable identity

$$(11.41)\qquad \mathfrak{R} = R\sqrt{-g} = \left(\sqrt{-g}\, g^{\sigma\rho}\begin{Bmatrix}\alpha\\ \sigma\ \alpha\end{Bmatrix}\right)_{|\rho} - \left(\sqrt{-g}\, g^{\sigma\rho}\begin{Bmatrix}\alpha\\ \sigma\ \rho\end{Bmatrix}\right)_{|\alpha} + \mathfrak{A}$$

The desired decomposition of $\mathfrak{R}$ into an ordinary divergence term and a simple expression involving only the $g_{\mu\nu}$ and their first derivatives is achieved.

Using Gauss's theorem for integrals of ordinary divergences, we now have

$$(11.42)\qquad J = \int_{D^4} \mathfrak{R}\, d^4x = \int_{D^4} \mathfrak{A}\, d^4x + \text{(surface terms)}$$

Let us then give a name to the first term of (11.42):

$$(11.43)\qquad H = \int_{D^4} \mathfrak{A}\, d^4x$$

Since $\mathfrak{A}$ is *not* a scalar density and depends on the choice of the coordinate system in a complicated manner, the integral H changes with different reference systems. However, given a specific coordinate system and a metric tensor $g_{\mu\nu}$ in it, we can assert that the values of J and H will undergo the same variation if we vary the metric tensor $g_{\mu\nu}$ in the coordinate region D^4 in an arbitrary way, but so that the variation of the $g_{\mu\nu}$ and their derivatives vanish on the boundary of D^4. Thus J and H have the same functional derivatives with respect to the metric tensor $g_{\mu\nu}$. This simple fact will be of great importance in what follows.

Let us next compute δH; since the integrand $\mathfrak{A}$ of H is a function of only the $g_{\mu\nu}$ and their first derivatives, we have, by the usual Euler-Lagrange method,

$$(11.44)\qquad \delta H = \delta \int_{D_4} \mathfrak{A}(g^{\mu\nu}, g^{\mu\nu}{}_{|\lambda})\, d^4x$$

$$= \int_{D_4}\left[\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}}\,\delta g^{\mu\nu} + \frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,\delta g^{\mu\nu}{}_{|\lambda}\right] d^4x$$

$$= \int_{D_4}\left[\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}} - \left(\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\right)_{|\lambda}\right]\delta g^{\mu\nu}\, d^4x + \text{(surface term)}$$

For variations which vanish on the boundary of D_4, the surface term will, of course, be zero; thus, equating δJ in (11.35) and δH in (11.44), we obtain

$$(11.45)\qquad \sqrt{-g}\, G_{\mu\nu} = \frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}} - \left(\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\right)_{|\lambda}$$

This very convenient and interesting expression for the Einstein tensor density is the net result of our variational analysis.

Our search for an expression $\sqrt{-g}\, t_\mu{}^\nu$ is now near an end. The final manipulation consists in relabeling indices in (11.45) and multiplying by $g^{\tau\beta}{}_{|\mu}$:

$$(11.46)\quad g^{\tau\beta}{}_{|\mu}(\sqrt{-g}\,G_{\tau\beta}) = \frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}}\,g^{\tau\beta}{}_{|\mu} - \left(\frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}{}_{|\lambda}}\right)_{|\lambda} g^{\tau\beta}{}_{|\mu}$$
$$= \frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}}\,g^{\tau\beta}{}_{|\mu} - \left(\frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}{}_{|\lambda}}\,g^{\tau\beta}{}_{|\mu}\right)_{|\lambda} + \frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}{}_{|\lambda}}\,g^{\tau\beta}{}_{|\lambda|\mu}$$

Observe that $\mathfrak{A}$ is a function of only $g^{\tau\beta}$ and $g^{\tau\beta}{}_{|\mu}$; thus

$$(11.47)\quad \mathfrak{A}_{|\mu} = \frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}}\,g^{\tau\beta}{}_{|\mu} + \frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}{}_{|\lambda}}\,g^{\tau\beta}{}_{|\lambda|\mu}$$

and (11.44) becomes

$$(11.48)\quad g^{\tau\beta}{}_{|\mu}(\sqrt{-g}\,G_{\tau\beta}) = \mathfrak{A}_{|\mu} - \left(\frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}{}_{|\lambda}}\,g^{\tau\beta}{}_{|\mu}\right)_{|\lambda}$$
$$= \left(\mathfrak{A}g_\mu{}^\lambda - \frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}{}_{|\lambda}}\,g^{\tau\beta}{}_{|\mu}\right)_{|\lambda}$$

By noting that $g^{\tau\beta}g_{\tau\lambda}$ is the invariant Kronecker delta, we see that

$$(11.49)\quad (g^{\tau\beta}g_{\tau\lambda})_{|\mu} = g^{\tau\beta}{}_{|\mu}g_{\tau\lambda} + g^{\tau\beta}g_{\tau\lambda|\mu} = 0$$

Thus

$$(11.50)\quad g^{\tau\beta}{}_{|\mu}G_{\tau\beta} = g^{\tau\beta}{}_{|\mu}g_{\tau\lambda}g_{\beta\sigma}G^{\lambda\sigma}$$
$$= -g^{\tau\beta}g_{\tau\lambda|\mu}g_{\beta\sigma}G^{\lambda\sigma} = -g_{\tau\beta|\mu}G^{\tau\beta}$$

so we can reverse the position of the indices on the left side of (11.48) if we reverse the sign:

$$(11.51)\quad \sqrt{-g}\,g_{\tau\beta|\mu}G^{\tau\beta} = -\left(\mathfrak{A}g_\mu{}^\lambda - \frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}{}_{|\lambda}}\,g^{\tau\beta}{}_{|\mu}\right)_{|\lambda}$$

This important result marks the end of our search, for if we define $\sqrt{-g}\,t_\mu{}^\nu$ to be

$$(11.52)\quad \sqrt{-g}\,t_\mu{}^\nu = \frac{1}{2C}\left(\mathfrak{A}g_\mu{}^\nu - \frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}{}_{|\nu}}\,g^{\tau\beta}{}_{|\mu}\right)$$

then, by using (11.51), we can write (11.18) as

$$(11.53)\quad (\sqrt{-g}\,T_\mu{}^\nu)_{|\nu} + (\sqrt{-g}\,t_\mu{}^\nu)_{|\nu} = (\sqrt{-g}\,[T_\mu{}^\nu + t_\mu{}^\nu])_{|\nu} = 0$$

Thus the quantity

$$(11.54)\quad \sqrt{-g}\,(T_\mu{}^\nu + t_\mu{}^\nu) = \sqrt{-g}\,T_\mu{}^\nu + \frac{1}{2C}\,\mathfrak{A}g_\mu{}^\nu - \frac{1}{2C}\,\frac{\partial\mathfrak{A}}{\partial g^{\tau\beta}{}_{|\nu}}\,g^{\tau\beta}{}_{|\mu}$$

has a zero ordinary divergence and represents the density of some conserved quantity. The expression $\sqrt{-g}\,t_\mu{}^\nu$ is usually referred to as the pseudo-tensor of the gravitational field.

The zero divergence of $\sqrt{-g}\,(T_\mu{}^\nu + t_\mu{}^\nu)$ gives rise to a conserved integral quantity in precisely the same way that the zero divergence of $T_\mu{}^\nu$ in flat space gives rise to the energy-momentum four-vector P_μ. Indeed, let us assume that the energy-momentum tensor is different from zero only in a finite part of space and that we can introduce a metric $g_{\mu\nu}$ which tends to the Lorentzian form as we approach infinity. It is evident that the quantity $t_\mu{}^\nu$ will tend to zero if we approach infinity. Let us now integrate (11.53) over all space-time between the hypersurfaces $x^0 = ct_i$ and $x^0 = ct_f$. As in the beginning of this section, we can apply integration by parts to obtain

$$(11.55)\quad \int_{t_f}\sqrt{-g}\,[T_\mu{}^0 + t_\mu{}^0]\,d^3x - \int_{t_i}\sqrt{-g}\,[T_\mu{}^0 + t_\mu{}^0]\,d^3x = 0$$

in analogy with (11.12). The quantity

$$(11.56)\quad P_\mu = \int_t \sqrt{-g}\,[T_\mu{}^0 + t_\mu{}^0]\,d^3x$$

is therefore conserved, and may be viewed as the *general relativistic generalization of the energy-momentum four-vector* of special relativity theory.

One must note carefully, however, that the quantity $\sqrt{-g}\,(T_\mu{}^\nu + t_\mu{}^\nu)$ is not a tensor and P_μ is not a generally covariant four-vector. This comes about, of course, because $\mathfrak{A}$ is not a scalar density. Furthermore, it should be noted that we assume that the coordinate system we are using when we integrate (11.53) over D^4 is Lorentzian at the spatial infinity of each coordinate; if this were not true, we could not obtain (11.55). For instance, *in polar coordinates*, (11.55) does *not* follow from (11.53).

Lastly, let us mention that our expression for $t_\mu{}^\nu$ is not unique; many authors use quite different expressions from ours and of course obtain results consistent with these alternative expressions.

It would be tempting to label $t_\mu{}^\nu$ as the energy-momentum tensor of the gravitational field; the remarks of the preceding paragraph, however, indicate that it is not possible to do this in a coordinate-invariant manner. It appears that the intimate connection between geometry, the gravitational field, and the notion of density makes the idea of the energy-momentum density of the gravitational field intrinsically noncovariant.

At this point we can summarize the difficulty of the situation as follows: Since $t_\mu{}^\nu$ is zero in a Lorentzian metric, we may say that we evaluate the gravitational energy-momentum quantity by its deviation from Lorentzian character. We know, however, that we may always introduce locally geodesic coordinates, which, at any chosen point, cause the non-Euclidean character of the geometry to disappear and lead to zero gravitational energy–momentum at that point. Thus the choice of some specific coordinate system means a particular localization of gravitational energy and momentum. It is satisfactory that the energy-momentum balance always comes out in the same way in the large, but the local distribution is coordinate-dependent and cannot be described in an intrinsic coordinate-independent way.

11.3 An Alternative Approach to the Conservation Laws: Energy-Momentum of the Schwarzschild Field

In the previous section we obtained the gravitational-field pseudo-tensor $t_\mu{}^\nu$ by considering a variation of the metric-tensor field. It is also possible to obtain an expression for $t_\mu{}^\nu$ by considering a variation of the coordinate system instead of the metric field, and in some respects the analysis is simpler. This is what we shall do in this section.

Let us again consider the noninvariant quantity

$$(11.57)\qquad \begin{aligned} H &= \int \mathfrak{A}\, d^4x \\ \mathfrak{A} &= \sqrt{-g}\, g^{\sigma\rho}\left[\begin{Bmatrix}\alpha\\ \sigma\ \rho\end{Bmatrix}\begin{Bmatrix}\beta\\ \alpha\ \beta\end{Bmatrix} - \begin{Bmatrix}\alpha\\ \beta\ \rho\end{Bmatrix}\begin{Bmatrix}\beta\\ \alpha\ \sigma\end{Bmatrix}\right]\end{aligned}$$

Clearly, $\mathfrak{A}$ involves only the $g^{\mu\nu}$ and their first derivatives, which fact we made use of in the preceding section. For any change of the metric field, the variation of $\mathfrak{A}$ will be

$$(11.58)\qquad \delta\mathfrak{A} = \frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}}\,\delta g^{\mu\nu} + \frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,\delta g^{\mu\nu}{}_{|\lambda}$$

Among all possible variations of $g^{\mu\nu}$, we shall investigate those which are due to a small change of coordinates of the form

$$(11.59)\qquad \bar{x}^\alpha = x^\alpha + \epsilon\xi^\alpha$$

where ξ^α is an arbitrary function of the x^α, and ϵ is a small constant. This coordinate transformation leads to a new metric tensor

$$(11.60)\qquad \begin{aligned}\bar{g}^{\mu\nu} &= \frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{\partial\bar{x}^\nu}{\partial x^\beta}\,g^{\alpha\beta} = \left(\delta^\mu{}_\alpha + \epsilon\frac{\partial\xi^\mu}{\partial x^\alpha}\right)\left(\delta^\nu{}_\beta + \epsilon\frac{\partial\xi^\nu}{\partial x^\beta}\right)g^{\alpha\beta} \\ &= g^{\mu\nu} + \epsilon\left(\frac{\partial\xi^\mu}{\partial x^\alpha}\,g^{\alpha\nu} + \frac{\partial\xi^\nu}{\partial x^\beta}\,g^{\mu\beta}\right) + O(\epsilon^2)\end{aligned}$$

Thus the first-order variation of $g^{\mu\nu}$ is

$$(11.61)\qquad \delta g^{\mu\nu} = \epsilon\left(\frac{\partial\xi^\mu}{\partial x^\alpha}\,g^{\alpha\nu} + \frac{\partial\xi^\nu}{\partial x^\beta}\,g^{\mu\beta}\right) = \epsilon(\xi^\mu{}_{|\alpha}g^{\alpha\nu} + \xi^\nu{}_{|\beta}g^{\mu\beta})$$

Some care must be exercised in calculating $\delta g^{\mu\nu}{}_{|\lambda}$; by definition,

$$(11.62)\qquad \bar{g}^{\mu\nu}{}_{|\lambda} = \frac{\partial\bar{g}^{\mu\nu}}{\partial\bar{x}^\lambda} = \frac{\partial\bar{g}^{\mu\nu}}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial\bar{x}^\lambda}$$

From the transformation equation (11.59), we have

$$(11.63)\qquad \frac{\partial x^\alpha}{\partial\bar{x}^\lambda} = \delta^\alpha{}_\lambda - \epsilon\frac{\partial\xi^\alpha}{\partial x^\lambda} + O(\epsilon^2)$$

Substituting (11.60) and (11.63) in (11.62), we obtain

$$(11.64)\qquad \begin{aligned}\frac{\partial\bar{g}^{\mu\nu}}{\partial\bar{x}^\lambda} &= \frac{\partial}{\partial x^\alpha}\left[g^{\mu\nu} + \epsilon\left(\frac{\partial\xi^\mu}{\partial x^\tau}\,g^{\tau\nu} + \frac{\partial\xi^\nu}{\partial x^\beta}\,g^{\mu\beta}\right)\right]\left[\delta^\alpha{}_\lambda - \epsilon\frac{\partial\xi^\alpha}{\partial x^\lambda}\right] + O(\epsilon^2) \\ &= \frac{\partial g^{\mu\nu}}{\partial x^\lambda} + \epsilon\left[\frac{\partial}{\partial x^\lambda}\left(\frac{\partial\xi^\mu}{\partial x^\tau}\,g^{\tau\nu} + \frac{\partial\xi^\nu}{\partial x^\beta}\,g^{\mu\beta}\right) - \frac{\partial g^{\mu\nu}}{\partial x^\alpha}\frac{\partial\xi^\alpha}{\partial x^\lambda}\right] + O(\epsilon^2)\end{aligned}$$

Hence the variation of $g^{\mu\nu}{}_{|\lambda}$ is

$$(11.65)\qquad \begin{aligned}\delta g^{\mu\nu}{}_{|\lambda} &= \epsilon[(\xi^\mu{}_{|\tau}g^{\tau\nu} + \xi^\nu{}_{|\beta}g^{\mu\beta})_{|\lambda} - g^{\mu\nu}{}_{|\alpha}\xi^\alpha{}_{|\lambda}] \\ &= \epsilon[g^{\tau\nu}{}_{|\lambda}\xi^\mu{}_{|\tau} + g^{\mu\beta}{}_{|\lambda}\xi^\nu{}_{|\beta} - g^{\mu\nu}{}_{|\alpha}\xi^\alpha{}_{|\lambda} + \xi^\mu{}_{|\tau|\lambda}g^{\tau\nu} + \xi^\nu{}_{|\beta|\lambda}g^{\mu\beta}]\end{aligned}$$

The variation of $\sqrt{-g}$ is easily obtained using the general relation (11.30) and (11.61):

$$(11.66)\qquad \begin{aligned}\delta\sqrt{-g} &= \frac{-\sqrt{-g}}{2}\,g_{\mu\nu}\,\delta g^{\mu\nu} \\ &= -\frac{\epsilon}{2}\sqrt{-g}\,g_{\mu\nu}(\xi^\mu{}_{|\alpha}g^{\alpha\nu} + \xi^\nu{}_{|\beta}g^{\mu\beta}) = -\epsilon\sqrt{-g}\,\xi^\alpha{}_{|\alpha}\end{aligned}$$

The variations in $g^{\mu\nu}$, $g^{\mu\nu}{}_{|\lambda}$, and $\sqrt{-g}$ which we have calculated in (11.61), (11.65), and (11.66) are general and hold for any coordinate transformation (11.59). We wish to use these relations to calculate $\delta\mathfrak{A}$; to do this we shall first use them to establish a very interesting relation involving $\mathfrak{A}$, which will in turn allow us to obtain a very simple and elegant form for $\delta\mathfrak{A}$. Let us specify the ξ^α to be linear functions of the coordinates x^μ. Equation (11.59) is then a linear transformation of coordinates; under such a linear transformation the Christoffel symbols transform as tensors, as is evident from (2.5). Thus the expression $\mathfrak{A}$ is a scalar density under the restricted class of linear transformations, which is apparent from (11.57). Since $\mathfrak{A}$ is a scalar density under (11.59), it has a very simple variation, which is, in fact, due entirely to the variation of the factor $\sqrt{-g}$ under the transformation (11.59). Indeed, we see that

$$\delta\mathfrak{A} = \left(\frac{\mathfrak{A}}{\sqrt{-g}}\right)\delta\sqrt{-g} \tag{11.67}$$

since $\mathfrak{A}/\sqrt{-g}$ is a scalar invariant. From (11.66) we therefore have

$$\delta\mathfrak{A} = -\epsilon\xi^{\alpha}{}_{|\alpha}\mathfrak{A} \tag{11.68}$$

for a linear transformation (11.59).

On the other hand, we may also calculate the variation $\delta\mathfrak{A}$ for a linear transformation (11.59) by substituting the variations (11.61) and (11.65) into the general expression for $\delta\mathfrak{A}$ in (11.58); since in the present case ξ^α is a linear function and has zero second derivatives, we obtain

$$\begin{aligned}\delta\mathfrak{A} &= \frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}}\,\delta g^{\mu\nu} + \frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,\delta g^{\mu\nu}{}_{|\lambda} \\ &= 2\epsilon\left(\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}}\,\xi^{\mu}{}_{|\alpha}g^{\alpha\nu} + \frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,\xi^{\mu}{}_{|\alpha}g^{\alpha\nu}{}_{|\lambda}\right) - \epsilon\,\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,g^{\mu\nu}{}_{|\alpha}\xi^{\alpha}{}_{|\lambda}\end{aligned} \tag{11.69}$$

If we now compare (11.68) and (11.69) and note that the $\xi^{\alpha}{}_{|\beta}$ are arbitrary constants, we arrive at a remarkable differential identity for $\mathfrak{A}$:

$$\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}}\,g^{\alpha\nu} + \frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,g^{\alpha\nu}{}_{|\lambda} - \frac{1}{2}\,\frac{\partial\mathfrak{A}}{\partial g^{\beta\nu}{}_{|\alpha}}\,g^{\beta\nu}{}_{|\mu} = -\tfrac{1}{2}\mathfrak{A}g_{\mu}{}^{\alpha} \tag{11.70}$$

It should be carefully noted that this identity was obtained by using variational methods and a linear coordinate transformation, but the identity is quite general and is completely independent of any coordinate transformation used in its derivation.

It is clear that, by substituting the variations of $g^{\mu\nu}$ and $g^{\mu\nu}{}_{|\lambda}$ in (11.61) and (11.65) into the variation of $\mathfrak{A}$ in (11.58), we can compute $\delta\mathfrak{A}$ for an arbitrary ξ^α and indeed could have computed $\delta\mathfrak{A}$ without bothering to obtain the identity (11.70); we shall now see, however, that (11.70) greatly simplifies the form of $\delta\mathfrak{A}$ and is well worth the price of its derivation. The substitutions into (11.58) give

$$\begin{aligned}\delta\mathfrak{A} = 2\epsilon\xi^{\mu}{}_{|\alpha}&\left(\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}}\,g^{\alpha\nu} + \frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,g^{\alpha\nu}{}_{|\lambda} - \frac{1}{2}\,\frac{\partial\mathfrak{A}}{\partial g^{\beta\nu}{}_{|\alpha}}\,g^{\beta\nu}{}_{|\mu}\right) \\ &+ 2\epsilon\,\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,\xi^{\mu}{}_{|\tau|\lambda}g^{\tau\nu}\end{aligned} \tag{11.71}$$

The straightforward substitution of (11.70) then gives the much simpler result

$$\delta\mathfrak{A} = -\epsilon\mathfrak{A}\xi^{\alpha}{}_{|\alpha} + 2\epsilon\,\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,\xi^{\mu}{}_{|\tau|\lambda}g^{\tau\nu} \tag{11.72}$$

which is quite general and holds for any coordinate variation (11.59).

Having obtained these formal results, let us return to physics. For a variation of the metric-tensor field which vanishes on the boundary of D^4, we found in the preceding section that $\delta J = \delta H$; J and H are defined in (11.20) and (11.43). If we choose the ξ^α in (11.59) to be zero and have zero first and second derivatives on the boundary of D^4, it is clear from (11.59) that $\delta g^{\mu\nu} = \delta g^{\mu\nu}{}_{|\lambda} = 0$ on the boundary of D^4, so $\delta H = \delta J$. However, we know that J is a scalar invariant, and thus must have a zero variation under any coordinate variation; thus, for this special coordinate variation,

$$\delta J = \delta H = \delta\int_{D^4}\mathfrak{A}\,d^4x = 0 \tag{11.73}$$

We may write this more conveniently by taking note of the fact that $\sqrt{-g}\,d^4x$ is a scalar invariant and has zero variation under a coordinate variation; then, since the range of the old and the new variables is the same, namely D^4, we have

$$\delta H = \delta\int_{D^4}\left(\frac{\mathfrak{A}}{\sqrt{-g}}\right)\sqrt{-g}\,d^4x = \int_{D^4}\delta\left(\frac{\mathfrak{A}}{\sqrt{-g}}\right)\sqrt{-g}\,d^4x = 0 \tag{11.74}$$

The variation of $\mathfrak{A}/\sqrt{-g}$ is easily obtained from the variation of $\mathfrak{A}$ in (11.72) and the variation of $\sqrt{-g}$ in (11.66):

$$\begin{aligned}\sqrt{-g}\,\delta\left(\frac{\mathfrak{A}}{\sqrt{-g}}\right) &= \delta\mathfrak{A} + \sqrt{-g}\,\mathfrak{A}\delta\left(\frac{1}{\sqrt{-g}}\right) \\ &= \delta\mathfrak{A} - \frac{\mathfrak{A}}{\sqrt{-g}}\,\delta\sqrt{-g} = 2\epsilon\,\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,\xi^{\mu}{}_{|\tau|\lambda}g^{\tau\nu}\end{aligned} \tag{11.75}$$

Thus the null variation of H implies

$$\delta H = 2\epsilon\int_{D^4}\frac{\partial\mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}}\,\xi^{\mu}{}_{|\tau|\lambda}g^{\tau\nu}\,d^4x = 0 \tag{11.76}$$

Since ξ^μ and $\xi^{\mu}{}_{|\tau}$ vanish on the boundary of D^4, we may integrate by parts twice to obtain an equivalent expression

$$\delta H = 2\epsilon \int_{D^4} \left(\frac{\partial \mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}} g^{\tau\nu} \right)_{|\tau|\lambda} \xi^\mu \, d^4x = 0 \tag{11.77}$$

Since ξ^μ is arbitrary inside D^4, we therefore obtain, finally, the divergence law

$$\left(\frac{\partial \mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}} g^{\tau\nu} \right)_{|\tau|\lambda} = 0 \tag{11.78}$$

It is thus apparent that the expression

$$\sqrt{-g}\, F_\mu{}^\tau = \left(\frac{\partial \mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}} g^{\tau\nu} \right)_{|\lambda} \tag{11.79}$$

is associated with some conserved physical quantity.

Let us work out $F_\mu{}^\tau$ explicitly to see what sort of physical quantity it represents:

$$\sqrt{-g}\, F_\mu{}^\tau = \left(\frac{\partial \mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}} \right)_{|\lambda} g^{\tau\nu} + \frac{\partial \mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}} g^{\tau\nu}{}_{|\lambda} \tag{11.80}$$

By solving the useful identity (11.70) for $(\partial\mathfrak{A}/\partial g^{\mu\nu})_{|\lambda} g^{\tau\nu}{}_{|\lambda}$ and substituting the result into (11.80), we arrive at

$$\sqrt{-g}\, F_\mu{}^\tau = \left[\left(\frac{\partial \mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}} \right)_{|\lambda} - \frac{\partial \mathfrak{A}}{\partial g^{\mu\nu}} \right] g^{\tau\nu} + \frac{1}{2} \frac{\partial \mathfrak{A}}{\partial g^{\beta\nu}{}_{|\tau}} g^{\beta\nu}{}_{|\mu} - \tfrac{1}{2}\mathfrak{A} g_\mu{}^\tau \tag{11.81}$$

Finally, let us invoke Eq. (11.45), which says that the first bracket in (11.81) is precisely $-\sqrt{-g}\, G_{\mu\nu}$; this gives

$$\sqrt{-g}\, F_\mu{}^\tau = -\sqrt{-g}\, G_{\mu\nu} - \tfrac{1}{2}\mathfrak{A} g_\mu{}^\tau + \frac{1}{2} \frac{\partial \mathfrak{A}}{\partial g^{\beta\nu}{}_{|\tau}} g^{\beta\nu}{}_{|\mu} \tag{11.82}$$

or, by using the field equations (11.9),

$$\sqrt{-g}\, F_\mu{}^\tau = -C\sqrt{-g}\, T_\mu{}^\tau - \tfrac{1}{2}\mathfrak{A} g_\mu{}^\tau + \frac{1}{2} \frac{\partial \mathfrak{A}}{\partial g^{\beta\nu}{}_{|\tau}} g^{\beta\nu}{}_{|\mu} \tag{11.83}$$

This last form we immediately recognize as being essentially the same conserved quantity as we obtained in the preceding section; indeed, by comparing with (11.54), we see that

$$F_\mu{}^\tau = -C[T_\mu{}^\tau + t_\mu{}^\tau] \tag{11.84}$$

Our alternative method to find the pseudo-tensor $F_\mu{}^\tau$ with zero divergence is of great significance for the general mathematical approach to

field equations in general relativity. It should be observed that we carried out many operations without using the explicit dependence of $\mathfrak{A}$ on the metric tensor $g_{\mu\nu}$. In fact, suppose we had started with an arbitrary expression $\mathfrak{A} = A\sqrt{-g}$, which depends only on the $g_{\mu\nu}$ and their first derivatives and on other field quantities that are independent of the metric tensor. To simplify matters, let us even assume that A is an actual scalar, i.e., is invariant under any change of variables. All derivations from (11.58) to (11.81) would have remained valid. Only to pass from (11.81) to (11.82) did we use the identity (11.45), which is based on the fundamental variational formula (11.35). But formula (11.79) is already sufficient to construct from the scalar density $\mathfrak{A}$ a pseudo-tensor density $F_\mu{}^\tau\sqrt{-g}$ whose ordinary divergence is zero. Such a pseudo-tensor density can always be interpreted as connected with a conserved quantity.
section and discuss the identity (11.84). Although we obtain the same conserved quantity by varying the coordinates as we obtained previously by varying the metric-tensor field, the present approach has the virtue that the conserved $F_\mu{}^\tau$ itself appears as an ordinary divergence in (11.79). This fact allows a simplification of the conservation law in special cases. Indeed, by the general procedure of the last section, it is evident that

$$P_\mu = \frac{-1}{C} \int_{V^3} \left(\frac{\partial \mathfrak{A}}{\partial g^{\mu\nu}{}_{|\lambda}} g^{0\nu} \right)_{|\lambda} d^3x \qquad C = -\frac{8\pi\kappa}{c^2} \tag{11.85}$$

is a conserved quantity if V^3 is the entire space at a given time. In the special case that the metric is independent of time, we need not sum over $\lambda = 0$, so we can use Gauss's theorem in three dimensions to obtain the energy-momentum content of any finite region V^3,

$$P_\mu = \frac{-1}{C} \int_{S^2} \frac{\partial \mathfrak{A}}{\partial g^{\mu\nu}{}_{|j}} g^{0\nu} n_j \, dS \tag{11.86}$$

where n_j is a unit normal to the surface element of S^2, which is the boundary of V^3. We therefore have the interesting and curious result that the generalized energy momentum P_μ of a volume V^3 may be determined from the values of the metric-tensor field and its derivatives *on the surface of* V^3; the detailed behavior of the field inside V^3 is irrelevant.

To conclude this section we shall compute P_0 for a Schwarzschild field using Eq. (11.86). This calculation will serve to illustrate the physics and to clarify the usefulness and the limited validity of the concept of the generalized energy momentum P_μ. As we stressed at the end of the previous section, we must use coordinates for which the metric is Lorentzian at spatial infinity, so the standard form of the Schwarzschild metric in polar coordinates will not do; instead, we shall use the isotropic form of Sec. 6.2, expressed in terms of the coordinates of special relativity:

$$ds^2 = \frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2} c^2\,dt^2 - \left(1 + \frac{m}{2\rho}\right)^4 (dx^2 + dy^2 + dz^2) \tag{11.87}$$

This form is clearly Lorentzian at infinity, $\rho = \infty$. The metric tensor is therefore

$$g_{\mu\nu} = \begin{pmatrix} \dfrac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2} & 0 & 0 & 0 \\ 0 & -\left(1 + \dfrac{m}{2\rho}\right)^4 & 0 & 0 \\ 0 & 0 & -\left(1 + \dfrac{m}{2\rho}\right)^4 & 0 \\ 0 & 0 & 0 & -\left(1 + \dfrac{m}{2\rho}\right)^4 \end{pmatrix} \tag{11.88}$$

and its inverse is

$$g^{\mu\nu} = \begin{pmatrix} \dfrac{(1 + m/2\rho)^2}{(1 - m/2\rho)^2} & 0 & 0 & 0 \\ 0 & -\left(1 + \dfrac{m}{2\rho}\right)^{-4} & 0 & 0 \\ 0 & 0 & -\left(1 + \dfrac{m}{2\rho}\right)^{-4} & 0 \\ 0 & 0 & 0 & -\left(1 + \dfrac{m}{2\rho}\right)^{-4} \end{pmatrix} \tag{11.89}$$

Since the metric tensor $g^{\mu\nu}$ is diagonal, the quantity P_0 which we wish to calculate is, from (11.86),

$$P_0 = \frac{-1}{C} \int_{S^2} \frac{\partial \mathfrak{A}}{\partial g^{00}{}_{|j}} g^{00} n_j\,dS \tag{11.90}$$

For convenience we shall take S^2 to be the surface of a sphere of radius $\rho = R$.

The problem now is to compute the quantity $\partial\mathfrak{A}/\partial g^{00}{}_{|j}$, where $\mathfrak{A}$ is considered to be a function of $g^{\mu\nu}$ and its first derivatives. Consider the first term of

$$\mathfrak{A} = \sqrt{-g}\, g^{\sigma\rho} \left[\begin{Bmatrix} \alpha \\ \sigma\ \rho \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha\ \beta \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \beta\ \sigma \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha\ \rho \end{Bmatrix} \right] \tag{11.91}$$

This term may be rewritten, with the use of the definition of the Christoffel symbols and the symmetry of $g^{\mu\nu}$, as

$$\sqrt{-g}\, g^{\sigma\rho} \begin{Bmatrix} \alpha \\ \sigma\ \rho \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha\ \beta \end{Bmatrix} = \frac{\sqrt{-g}}{4} g^{\sigma\rho} g^{\alpha\tau} (g_{\rho\tau|\sigma} + g_{\sigma\tau|\rho} - g_{\sigma\rho|\tau}) g^{\beta\kappa} g_{\beta\kappa|\alpha} \tag{11.92}$$

In order to express this in terms of the derivatives of $g^{\mu\nu}$ instead of the derivatives of $g_{\mu\nu}$, we make use of the convenient elementary relation $g_{\mu\nu|\lambda} g^{\nu\epsilon} = -g^{\nu\epsilon}{}_{|\lambda} g_{\mu\nu}$ and obtain

$$\sqrt{-g}\, g^{\sigma\rho} \begin{Bmatrix} \alpha \\ \sigma\ \rho \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha\ \beta \end{Bmatrix} = \frac{\sqrt{-g}}{4} g^{\alpha\tau} (g_{\rho\tau} g^{\sigma\rho}{}_{|\sigma} + g_{\sigma\tau} g^{\sigma\rho}{}_{|\rho} - g_{\sigma\rho} g^{\sigma\rho}{}_{|\tau}) g_{\beta\kappa} g^{\beta\kappa}{}_{|\alpha} \tag{11.93}$$

This is easily differentiated with respect to $g^{00}{}_{|j}$:

$$\frac{\partial}{\partial g^{00}{}_{|j}} \left(\sqrt{-g}\, g^{\sigma\rho} \begin{Bmatrix} \alpha \\ \sigma\ \rho \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha\ \beta \end{Bmatrix} \right) = \frac{\sqrt{-g}}{2} g_{00} g^{j\tau} (g_{\rho\tau} g^{\sigma\rho}{}_{|\sigma} - g_{\sigma\rho} g^{\sigma\rho}{}_{|\tau}) \tag{11.94}$$

The second term of $\mathfrak{A}$ is also easily handled; note first that

$$\begin{aligned} \frac{\partial}{\partial g^{00}{}_{|j}} \left(\sqrt{-g}\, g^{\sigma\rho} \begin{Bmatrix} \alpha \\ \beta\ \sigma \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha\ \rho \end{Bmatrix} \right) &= \sqrt{-g}\, g^{\sigma\rho} \left(\begin{Bmatrix} \alpha \\ \beta\ \sigma \end{Bmatrix} \frac{\partial}{\partial g^{00}{}_{|j}} \begin{Bmatrix} \beta \\ \alpha\ \rho \end{Bmatrix} + \begin{Bmatrix} \beta \\ \alpha\ \rho \end{Bmatrix} \frac{\partial}{\partial g^{00}{}_{|j}} \begin{Bmatrix} \alpha \\ \beta\ \sigma \end{Bmatrix} \right) \\ &= 2\sqrt{-g}\, g^{\sigma\rho} \begin{Bmatrix} \alpha \\ \beta\ \sigma \end{Bmatrix} \frac{\partial}{\partial g^{00}{}_{|j}} \begin{Bmatrix} \beta \\ \alpha\ \rho \end{Bmatrix} \end{aligned} \tag{11.95}$$

This considerably simplifies the calculation. Next, using the definition of the Christoffel symbol, we write this as

$$\frac{\partial}{\partial g^{00}{}_{|j}} \left(\sqrt{-g}\, g^{\sigma\rho} \begin{Bmatrix} \alpha \\ \beta\ \sigma \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha\ \rho \end{Bmatrix} \right) \tag{11.96}$$

$$= \sqrt{-g} \begin{Bmatrix} \alpha \\ \beta \ \sigma \end{Bmatrix} \frac{\partial}{\partial g^{00}{}_{|j}} (-g_{\rho\tau} g^{\sigma\rho} g^{\beta\tau}{}_{|\alpha} - g_{\alpha\tau} g^{\sigma\rho} g^{\beta\tau}{}_{|\rho} + g_{\alpha\rho} g^{\beta\tau} g^{\sigma\rho}{}_{|\tau})$$

which may be immediately differentiated to give

$$\frac{\partial}{\partial g^{00}{}_{|j}} \left(\sqrt{-g}\, g^{\sigma\rho} \begin{Bmatrix} \alpha \\ \beta \ \sigma \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha \ \rho \end{Bmatrix} \right) = -\sqrt{-g} \begin{Bmatrix} j \\ 0 \ 0 \end{Bmatrix} \tag{11.97}$$

Since the metric is time-independent, this further simplifies to

$$\frac{\partial}{\partial g^{00}{}_{|j}} \left(\sqrt{-g}\, g^{\sigma\rho} \begin{Bmatrix} \alpha \\ \beta \ \sigma \end{Bmatrix} \begin{Bmatrix} \beta \\ \alpha \ \rho \end{Bmatrix} \right) = -\sqrt{-g} \begin{Bmatrix} j \\ 0 \ 0 \end{Bmatrix} \tag{11.98}$$
$$= -\frac{\sqrt{-g}}{2} g^{j\lambda} (g_{\lambda 0|0} + g_{\lambda 0|0} - g_{00|\lambda}) = \frac{\sqrt{-g}}{2} g^{j\lambda} g_{00|\lambda}$$

Combining (11.98) and (11.94), we finally have

$$\frac{\partial \mathfrak{A}}{\partial g^{00}{}_{|j}} = \frac{\sqrt{-g}}{2} [g_{00} g^{\sigma j}{}_{|\sigma} - g_{00} g^{j\tau} g_{\sigma\rho} g^{\sigma\rho}{}_{|\tau} - g^{j\sigma} g_{00|\sigma}] \tag{11.99}$$

This is a general result valid for any time-independent metric.

For the specific metric tensor (11.89) which is diagonal and spatially isotropic, we obtain from (11.99) after an easy rearrangement of terms

$$g^{00} \frac{\partial \mathfrak{A}}{\partial g^{00}{}_{|j}} = -\sqrt{-g}\, g^{11}{}_{|j} \tag{11.100}$$

Using the explicit form (11.89) for $g^{\mu\nu}$, we have, finally,

$$g^{00} \frac{\partial \mathfrak{A}}{\partial g^{00}{}_{|j}} = \frac{2mx^j}{\rho^3} \left(1 - \frac{m}{2\rho}\right) \tag{11.101}$$

This is the exact integrand of (11.90) in explicit form; we need only integrate over a sphere of radius R to find the total "energy" P_0 inside the sphere. We thus have to calculate the integral (11.90) in the form

$$P_0 = \frac{c^2}{8\pi\kappa} \int_{S^2} 2m \frac{1}{\rho^3} x^j n_j \left(1 - \frac{m}{2\rho}\right) dS \tag{11.102}$$

One should now beware of an error which is frequently committed. One should not evaluate the surface element dS in the Schwarzschild metric, but in the Euclidean metric of the marker space of the x^j. Indeed, we obtained the surface integral (11.86) from the volume integral (11.85) by use of the Gauss integral theorem, and n_j and dS are to be understood in the sense required by classical integral calculus. Hence dS is the ordinary Euclidean surface element of a sphere S^2 with radius $\rho = R$. Clearly, $x^j n_j = R$, and (11.102) reduces to

$$P_0 = \frac{mc^2}{\kappa} \left(1 - \frac{m}{2R}\right) \tag{11.103}$$

Let us replace the mass parameter m by the usual mass M of the source of the Schwarzschild field as given by (6.54). We then find that

$$P_0 = M \left(1 - \frac{\kappa M}{2c^2 R}\right) \tag{11.104}$$

As we might expect, the asymptotic value of P_0 is M, the total mass of the particle or spherical body, in agreement with special relativity theory. For finite values of R, this formula would give us the gravitational energy content within a sphere of radius R. Interestingly enough, the energy of gravitation starts with the value zero for $R = m/2$; that is, the entire mass is due to the contributions of the gravitational field outside of the radius $R = m/2$ in the isotropic Schwarzschild line element. From this critical radius the energy content increases with the radius of the sphere. At the distance R from the center of the field we have an energy density

$$\Delta P_0 = \frac{\kappa M^2}{8\pi R^4 c^2} \tag{11.105}$$

This result stands in complete analogy to the classical formula for the electrical energy density around a charged singularity with total charge e. In our units where energy is measured as mass, we have, for the corresponding energy density,

$$\Delta P_0 = \frac{e^2}{8\pi R^4 c^2} \tag{11.106}$$

The absence of a κ constant in the electrical energy density term is due to the choice of the units for the electrical charge such that no constant κ occurs in the Coulomb law. However, the same distance dependence in Newton's and Coulomb's laws of attraction leads to the same radial dependence of the energy density.

11.4 Variational Principles in General Relativity Theory: A Lagrangian Density for the Gravitational Field

The formal considerations of the preceding sections are relevant to a variational formulation of various physical theories within the framework of general relativity. In this section we shall first show that the equations of the gravitational field in empty space can easily be expressed in a variational form: The Lagrangian of this variational principle has in fact been obtained in the preceding sections. Then we shall show how the Lagrangian for the gravitational field in empty space can be extended in such a way that the resultant variational principle will describe also the effect of electromagnetic fields. Our considerations will be illustrative of the great flexibility and power of the variational method and will suggest the possibility of numerous applications in various fields of physics.

We begin with Eqs. (11.20) and (11.35) of Sec. 11.2, which together imply

(11.107) $$\delta J = \delta \int_{D^4} R \sqrt{-g}\, d^4x = \int_{D^4} G_{\mu\nu} \sqrt{-g}\, \delta g^{\mu\nu}\, d^4x$$

Since the equations of the gravitational field in empty space are $G_{\mu\nu} = 0$, we see that we may express the 10 field equations by the single variational principle

(11.108) $$\delta J = \delta \int_{D^4} R \sqrt{-g}\, d^4x = 0$$

The variations admitted in (11.108) are such that the metric tensor and its first derivatives do not vary on the boundary of D^4.

We may interpret (11.107) to mean that the Einstein tensor density $G_{\mu\nu} \sqrt{-g}$ is the variational derivative of the scalar density $R \sqrt{-g}$ under a variation of the metric tensor. Furthermore, we see that $R \sqrt{-g}$ can be interpreted as the Lagrangian density of the gravitational field in empty space. In Sec. 10.4 we have shown that the field equations of gravitation in nonempty space may be written as in (10.80)

(11.109) $$G_{\mu\nu} = CT_{\mu\nu} \qquad C = -\frac{8\pi\kappa}{c^2}$$

The appearance of the Einstein tensor in these more general equations and our above results suggest the possibility of extending the variational formulation (11.108) to nonempty space.

Consider a physical system characterized by a specific energy-momentum tensor $T_{\mu\nu}$. Our aim is then to construct a scalar density $L \sqrt{-g}$,

which depends on the metric tensor $g^{\mu\nu}$ and possibly other field variables (electromagnetic potentials, velocity fields, etc.) such that

(11.110) $$\delta \int_{D^4} L \sqrt{-g}\, d^4x = \int_{D^4} T_{\mu\nu} \sqrt{-g}\, \delta g^{\mu\nu}\, d^4x$$

under a variation of the metric field as considered above. If we could do this, the gravitational field equations in nonempty space (11.109) could be expressed in the variational form

(11.111) $$\delta \int_{D^4} (R - CL) \sqrt{-g}\, d^4x = 0$$

where only the metric-tensor field is varied. The scalar density $(R - CL) \sqrt{-g}$ would represent the Lagrangian density of the extended system.

Observe, however, that Eqs. (11.109), or equivalently (11.111), do not completely describe the physical system, but only its gravitational aspects. It is often possible to choose the expression L in such a way that the variation of the same integral as in (11.111) with respect to the additional field variables associated with $T_{\mu\nu}$ vanishes as a consequence of the additional field equations for these variables. In this case the unrestricted variation principle (11.111) would give a complete description of the physical system. We should then consider $(R - CL) \sqrt{-g}$ as the Lagrangian density of the entire system and $R \sqrt{-g}$ as the contribution of the gravitational field to the total Lagrangian.

We illustrate these general remarks by an example which is of considerable importance in its own right, the electromagnetic field. Let $F_{\mu\nu}$ be the antisymmetric tensor which describes the electromagnetic field (as discussed in Secs. 4.1 and 9.1) and define the scalar

(11.112) $$L = AF_{\mu\nu}F^{\mu\nu} = Ag^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$$

with a constant A, which will be conveniently prescribed later. In this case we can easily calculate the variation of the integral (11.110) by the use of (11.30) and find, considering $F_{\mu\nu}$ independent of $g_{\mu\nu}$,

(11.113) $$\delta \int_{D^4} L \sqrt{-g}\, d^4x = -2A \int_{D^4} [F_\mu{}^\rho F_{\rho\nu} + \tfrac{1}{4} F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}] \sqrt{-g}\, \delta g^{\mu\nu}\, d^4x$$

On the other hand, we showed in (10.69) that the energy-momentum tensor of the electromagnetic field in empty space has the form

$$T_{\mu\nu} = F_{\mu}{}^{\rho}F_{\rho\nu} + \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu} \tag{11.114}$$

Comparison of (11.113) and (11.114) suggests the choice $A = -\frac{1}{2}$, in which case (11.113) agrees precisely with (11.110). Thus one possible form of the Lagrangian density of the combined gravitational and electromagnetic field in matter-free space has been established to be

$$(R - CL)\sqrt{-g} = \left(R + \frac{C}{2}F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g} \tag{11.115}$$

Surprisingly enough, the same Lagrangian density (11.115) which leads to the Einstein field equations of gravitation if we vary the metric tensor leads also to the Maxwell field equations of electromagnetism if we vary the potentials of the electromagnetic field. To demonstrate this statement, we remind the reader that one set of Maxwell equations, namely,

$$\{F_{\mu\nu|\lambda}\} = 0 \tag{11.116}$$

expresses the fact that $F_{\mu\nu}$ is a closed tensor (Sec. 4.1) and that it possesses a vector potential ϕ_μ. That is, $F_{\mu\nu}$ may be written in the form

$$F_{\mu\nu} = \phi_{\mu|\nu} - \phi_{\nu|\mu} \tag{11.117}$$

Conversely, if we introduce an arbitrary four-vector ϕ_μ and *define* $F_{\mu\nu}$ by (11.117), the set of equations (11.116) will be automatically fulfilled. We consider, therefore, the vector potential ϕ_μ as the independent field variable which must obey only the remaining set of Maxwell equations (4.63) in empty space,

$$F^{\mu\nu}{}_{\|\nu} = \frac{1}{\sqrt{-g}}(F^{\mu\nu}\sqrt{-g})_{|\nu} = 0 \tag{11.118}$$

We shall show now that these equations are precisely the Euler-Lagrange equations of the variational problem (11.111) for the Lagrangian (11.115) under a variation of the potential ϕ_μ. Indeed, observe that $R\sqrt{-g}$ is independent of the vector potential ϕ_μ; so from (11.115) we obtain the Euler-Lagrange equations

$$-\left[\frac{\partial(L\sqrt{-g})}{\partial\phi_{\mu|\nu}}\right]_{|\nu} = 2C(F^{\mu\nu}\sqrt{-g})_{|\nu} = 0 \tag{11.119}$$

which are identical with (11.118) as asserted.

In summary, we have shown that the combined Einstein and Maxwell field equations in matter-free space can be condensed into the single

variational principle

$$\delta\int_{D^4}\left(R + \frac{C}{2}F_{\alpha\beta}F^{\alpha\beta}\right)\sqrt{-g}\,d^4x = 0 \tag{10.135}$$

We obtain Einstein's equations if we vary the metric potentials $g_{\mu\nu}$, and Maxwell's equations for empty space if we vary the electromagnetic potentials ϕ_μ. The variational condition (11.120) summarizes all differential equations of the theory at points of space-time where no particles are located.

Lastly, let us remark that it is possible to extend the above variational principle to include systems with charges and masses. It is well known from special relativity that the Maxwell equations can be derived from a variational principle with a Lagrangian proportional to

$$L = \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + 2\phi_\mu s^\mu \tag{11.121}$$

Here s^μ is the current four-vector (Sec. 4.1). The ϕ_μ are considered as independent variables and $F_{\mu\nu}$ is defined by (11.117). It is immediate that the variational problem

$$\delta\int_{D^4} L\sqrt{-g}\,d^4x = 0 \tag{11.122}$$

leads to the Euler-Lagrange equations

$$\left[\frac{\partial(L\sqrt{-g})}{\partial\phi_{\mu|\nu}}\right]_{|\nu} - \frac{\partial(L\sqrt{-g})}{\partial\phi_\mu} = (F^{\mu\nu}\sqrt{-g})_{|\nu} - s^\mu\sqrt{-g} = 0 \tag{11.123}$$

which are identical with the Maxwell equations (4.63) even in the case of general relativity theory. However, the contribution of the masses and their velocities which must be associated with the charges requires additional terms in the complete Lagrangian. We shall not enter into the laborious discussion of the entire system of gravitational fields, electromagnetic fields, charges, and masses.

In this section we have shown that a combined theory of the gravitational and electromagnetic fields in matter-free space can be expressed in very elegant form. However, we wish to emphasize that this is only a demonstration of the power of the formal mathematical tools and by no means a step in the direction of a unified field theory. The aim of a unified field theory is to imbed electromagnetic phenomena into geometry in a way analogous to that done for gravitation. The discussion of this problem will be taken up in Chap. 15.

11.5 The Scalar Tensor Variation of Relativity Theory

We may use the mathematical developments of the preceding sections to study an interesting variant of general relativity theory. The problem of understanding the numerical value of a given physical constant has always intrigued physicists. Indeed one may consider the reduction of the number of arbitrary or unrelated physical constants to be a measure of the overall progress of physics. The puzzling aspect of arbitrary physical constants is especially evident in the case of the gravitational constant, $\kappa = 6.67 \times 10^{-8}$ dyne-cm/g^2, which is an extraordinarily small number; to state this in a way that is independent of the arbitrary units adopted by experimentalists we note that the gravitational attraction between two electrons at rest is less than the electrostatic repulsion by a factor of about 4×10^{42}. In an attempt to understand the size of κ, among other things, Brans and Dicke (1961) have developed a theory in which κ is considered to be related to a new scalar field which is determined by the distribution of mass-energy in the universe. This is in accord with the ideas of Mach, who felt that the bulk material of the universe should somehow determine the inertial and hence the gravitational properties of individual bodies.

Let us motivate the introduction of the scalar field in the Brans-Dicke theory by noting an interesting numerical relation; as we shall discuss in Sec. 12.1, the characteristic "size" of the universe is about $R = 10^{10}$ light years, and its average density is very roughly 10^{-31} g/cm^3, with an uncertainty of several orders of magnitude. This leads to the very rough numerical relation

$$\frac{\kappa}{c^2}\frac{M}{R} \sim 1 \tag{11.124}$$

where M is the total mass of the universe (see Exercise 11.3). This may also be expressed as

$$\frac{1}{\kappa} \sim \frac{M}{c^2 R} \tag{11.124'}$$

The form of this relation suggests that $1/\kappa$ may be equated with a scalar field φ which is itself determined by a Poissonlike equation with the bulk matter density of the universe as source. That is,

$$\nabla^2\varphi \sim \frac{\rho}{c^2} \tag{11.125}$$

This clearly has $\varphi \sim M/Rc^2$ as a solution.

It is very easy to make these considerations precise and covariant. We replace $\nabla^2\varphi$ by $\varphi_{|\mu\|\nu}g^{\mu\nu}$ and the density ρ by the scalar $T^\alpha{}_\alpha$ and write

$$\varphi^{\|\mu}{}_{\|\mu} = \varphi_{\|\mu\|\nu}g^{\mu\nu} = \frac{4\pi\lambda T^\alpha{}_\alpha}{c^2} \qquad \varphi_{|\mu} = \varphi_{\|\mu} \tag{11.126}$$

The constant λ is a new dimensionless coupling constant. We anticipate that this new constant, which replaces κ as the fundamental constant of gravitational theory, will be of order unity, in accord with (11.125). That is, we adopt the attitude that a dimensionless constant of order unity is more "natural" or easily accepted than the dimensional constant κ, which is small in the sense noted above. Of course it is the task of experimental or observational physics, assuming the correctness of the theory, to determine the value of λ using the further development and predictions of the theory.

Equation (11.126) is the first fundamental equation of the Brans-Dicke theory. To preserve the structure of conventional relativity theory as much as possible we shall retain the general relativistic interpretation of the metric as determining the trajectories of test bodies in a curved Riemannian manifold; the φ field is to have no direct influence on such motion. This implies, for example, that $T^{\mu\nu}$ will be divergenceless, as discussed in Sec. 11.1. It thus remains only to obtain field equations for the metric tensor. We approach this problem by suitably generalizing the Lagrangian formulation of general relativity contained in (11.111). We rewrite it, with κ in evidence, as

$$\delta\left(\frac{J}{\kappa}\right) = \delta\int\left(\frac{1}{\kappa}R + \frac{8\pi}{c^2}L\right)\sqrt{-g}\,d^4x = 0 \tag{11.127}$$

The substitution for the constant $1/\kappa$ of the field φ is the obvious way to generalize the first term; the second term needs no modification, and L will be defined as before in (11.110). To include the dynamics of the scalar field we must add a suitable Lagrangian for φ. In special relativity it is easy to show that an appropriate Lagrangian is proportional to $\varphi_{|\alpha}\varphi_{|\beta}g^{\alpha\beta}$ (Exercise 11.4). In order to add such a term to the above Lagrangian without introducing a new dimensional constant we are led to

$$\delta J = \delta\int\left[\varphi R + \frac{8\pi}{c^2}L + \omega\frac{\varphi_{|\alpha}\varphi_{|\beta}}{\varphi}g^{\alpha\beta}\right]\sqrt{-g}\,d^4x = 0 \tag{11.128}$$

where ω is a dimensionless constant. We shall presently relate it to λ.

This generalization of (11.127) is clearly dimensionally consistent and evidently as simple as possible.

The variational problem (11.128) with φ considered as an independent field leads immediately to the Euler-Lagrange equations; using formula (3.12) for the divergence, we obtain

$$\frac{-2\omega}{\varphi}\,\varphi^{\|\alpha}{}_{\|\alpha} + \frac{\omega}{\varphi^2}\,\varphi_{\|\alpha}\varphi^{\|\alpha} + R = 0 \tag{11.129}$$

We next must vary $g^{\mu\nu}$ to obtain the remaining field equations. From (11.128), using (11.32) and (11.110), we obtain easily

$$\begin{aligned}\delta J = \int \Big[\varphi G_{\mu\nu} &+ \frac{8\pi}{c^2}\,T_{\mu\nu} + \frac{\omega}{\varphi}\,\varphi_{|\mu}\varphi_{|\nu} \\ &- \frac{\omega}{2\varphi}\,g_{\mu\nu}\varphi^{\|\alpha}\varphi_{|\alpha}\Big]\sqrt{-g}\,\delta g^{\mu\nu}\,d^4x \\ &+ \int \varphi g^{\mu\nu}\left[\delta\left\{ {\alpha \atop \mu\ \alpha} \right\}_{\|\nu} - \delta\left\{ {\alpha \atop \mu\ \nu} \right\}_{\|\alpha}\right]\sqrt{-g}\,d^4x\end{aligned} \tag{11.130}$$

Only the last two terms, which we shall label $\delta\mathfrak{M}$ and $\delta\mathfrak{N}$ must be further simplified to obtain a field equation linking $G_{\mu\nu}$ to $T_{\mu\nu}$ and φ, analogous to the Einstein equations. To simplify $\delta\mathfrak{M}$ we use (11.34) and integrate by parts, again using the divergence formula (3.12),

$$\begin{aligned}\delta\mathfrak{M} &= \int \varphi\left(g^{\mu\nu}\,\delta\left\{ {\alpha \atop \mu\ \alpha} \right\}\right)_{\|\nu}\sqrt{-g}\,d^4x \\ &= \int \varphi\left(\sqrt{-g}\,g^{\mu\nu}\delta\left\{ {\alpha \atop \mu\ \alpha} \right\}\right)_{|\nu} d^4x \\ &= -\int \varphi_{|\nu}g^{\mu\nu}\delta\left\{ {\alpha \atop \mu\ \alpha} \right\}\sqrt{-g}\,d^4x\end{aligned} \tag{11.131}$$

Finally we use (11.14), (11.30), and the fact that the variation operation and ordinary differentiation commute to obtain

$$\begin{aligned}\delta\mathfrak{M} &= -\int \varphi_{|\nu}g^{\mu\nu}\sqrt{-g}\,\delta\,(\log\sqrt{-g})_{|\mu}\,d^4x \\ &= \int (\varphi_{|\nu}g^{\mu\nu}\sqrt{-g})_{|\mu}\,\delta(\log\sqrt{-g})\,d^4x \\ &= -\tfrac{1}{2}\int \varphi^{\|\nu}{}_{\|\mu}g_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x\end{aligned} \tag{11.132}$$

which we see by reference to (11.130) is in convenient form.

The last term $\delta\mathfrak{N}$ involves a bit more algebra than the above but is straightforward. Proceeding as with $\delta\mathfrak{M}$, we simplify it to

$$\delta\mathfrak{N} = \int \varphi_{|\alpha}g^{\mu\nu}\delta\left\{ {\alpha \atop \mu\ \nu} \right\}\sqrt{-g}\,d^4x \tag{11.133}$$

We observe that, analogous to the product rule for differentiation,

$$g^{\mu\nu}\delta\left\{ {\alpha \atop \mu\ \nu} \right\} = g^{\mu\nu}\,\delta g^{\alpha\sigma}[\mu\nu,\sigma] + g^{\mu\nu}g^{\alpha\sigma}\delta[\mu\nu,\sigma] \tag{11.134}$$

If we substitute this into (11.133) and integrate the last term by parts, the result is

$$\begin{aligned}\delta\mathfrak{N} = \int \Big[\varphi_{|\alpha}g^{\mu\nu}[\mu\nu,\sigma]\,\delta g^{\alpha\sigma} &- \varphi^{|\sigma}{}_{|\nu}g^{\mu\nu}\,\delta g_{\mu\sigma} - \varphi^{|\sigma}g^{\mu\nu}{}_{|\nu}\,\delta g_{\mu\sigma} \\ &- \varphi^{|\sigma}g^{\mu\nu}\left\{ {\alpha \atop \alpha\ \nu} \right\}\delta g_{\mu\sigma} + \tfrac{1}{2}\varphi^{\|\sigma}{}_{\|\sigma}g^{\mu\nu}\,\delta g_{\mu\nu} \\ &+ \tfrac{1}{2}\varphi^{|\sigma}g^{\mu\nu}{}_{|\sigma}\,\delta g_{\mu\nu}\Big]\sqrt{-g}\,d^4x\end{aligned} \tag{11.135}$$

To put this in a form analogous to (11.130) and (11.132) we use (11.28) and the following relations, which follow from the constancy of $g_{\mu\nu}g^{\nu\tau} = \delta_\mu{}^\tau$:

$$\begin{aligned}g^{\mu\nu}\,\delta g_{\mu\sigma} &= -g_{\mu\sigma}\,\delta g^{\mu\nu} \qquad \delta g_{\mu\nu} = -g_{\mu\alpha}g_{\gamma\beta}\,\delta g^{\alpha\beta} \\ g^{\omega\tau}{}_{|\mu}g_{\omega\sigma} &= -g^{\omega\tau}g_{\omega\sigma|\mu}\end{aligned} \tag{11.136}$$

This leads to the final covariant result for $\delta\mathfrak{N}$

$$\delta\mathfrak{N} = \int[-\tfrac{1}{2}\varphi^{\|\sigma}{}_{\|\sigma}g_{\mu\nu} + \varphi_{\|\mu\|\nu}]\,\delta g^{\mu\nu}\sqrt{-g}\,d^4x \tag{11.137}$$

We can now combine (11.130), (11.132), and (11.137) to obtain the following replacement for the Einstein field equations

$$\begin{aligned}R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = G_{\mu\nu} = -\frac{8\pi}{\varphi c^2}\,T_{\mu\nu} &- \frac{\omega}{\varphi^2}\,(\varphi_{|\mu}\varphi_{|\nu} - \tfrac{1}{2}g_{\mu\nu}\varphi^{|\alpha}\varphi_{|\alpha}) \\ &+ \frac{1}{\varphi}\,(\varphi_{\|\mu\|\nu} - g_{\mu\nu}\varphi^{\|\sigma}{}_{\|\sigma})\end{aligned} \tag{11.138}$$

This equation also allows us to obtain the scalar R in terms of the scalars T and φ

$$R = \frac{8\pi}{c^2\varphi} T - \frac{\omega}{\varphi^2} \varphi_{|\mu}\varphi^{|\mu} - \frac{3}{\varphi} \varphi^{\|\alpha}{}_{\|\alpha} \tag{11.139}$$

Finally, substituting this into (11.129), we obtain a more useful form for the scalar field equation, which in fact is precisely that anticipated in (11.126)

$$\varphi^{\|\alpha}{}_{\|\alpha} = \frac{8\pi T}{c^2(3 + 2\omega)} \tag{11.140}$$

where we identify $\lambda = 2/(3 + 2\omega)$.

Equations (11.138) and (11.140) constitute a complete basis for the Brans-Dicke theory. One may investigate all the usual problems of general relativity theory, such as those discussed in Chap. 6, and make predictions which will differ from those of conventional general relativity theory, thereby providing a means of testing the theory and measuring the constant ω. The reader is referred to the problems at the end of this chapter and to the bibliography for details of the specific problems. Although we shall not discuss the specific predictions of the Brans-Dicke theory in detail, we note that in the limit of large ω or small λ (11.140) gives

$$\varphi^{\|\mu}{}_{\|\mu} = 0 \qquad \varphi = \text{const} = \langle\varphi\rangle \tag{11.141}$$

and so (11.138) becomes the usual Einstein equation with $\kappa = 1/\langle\varphi\rangle$. That is, the theory goes over in this limit to conventional general relativity (see also Exercise 11.5).

In order to obtain a lower estimate for ω we mention one specific prediction of the Brans-Dicke theory. In Chap. 6 we discussed the perihelion precession of Mercury and Dicke's suggestion that roughly 8 per cent is due to the quadrupole moment of the sun, leaving only 40″ to be explained in terms of relativistic effects. The Brans-Dicke theory predicts for this shift $(3\omega + 4)/(3\omega + 6)$ times the Einstein value. Thus the Brans-Dicke theory is in agreement with this result if we set $(3\omega + 4)/(3\omega + 6) = 0.92$, which implies $\omega \cong 6.2$, or $\lambda \cong 0.13$. Of course the quadrupole effect remains an unsettled question, as noted in Chap. 6. At present the other observational tests of relativity discussed in Chap. 6 are not capable of distinguishing between general relativity and the scalar tensor, but improvements in accuracy should provide definitive tests in the near future. Such tests will give a specific value for ω, or if they are consistent with general relativity, they will give a lower bound on ω. It is important to note that the existence of the scalar field cannot be disproved since general relativity is the limit case for $\omega = \infty$: it cannot be decided by observation whether ω is very large or infinite (Prob. 11.7).

Exercises

11.1 Discuss further the criteria for a test body noted in Sec. 11.1. In particular study how large a test body may be before gravitational tidal forces become appreciable and the body ceases to act as if it had zero size.

11.2 Verify (11.119).

11.3 The argument used to motivate the introduction of the scalar φ in the Brans-Dicke theory depends on the relation (11.124). However, a similar relation can be obtained as a consequence of conventional relativity theory, as we shall see in Chap. 13. Does this weaken the motivation for the scalar tensor theory?

11.4 Show that in special relativity a Lagrangian proportional to $\varphi_{|\alpha}\varphi_{|\beta}g^{\alpha\beta}$ leads to the wave equation $\Box^2\varphi = 0$ and is thus appropriate to a scalar field.

11.5 For large ω show that the Brans-Dicke equations (11.138) and (11.140) become

$$\varphi = \langle\varphi\rangle + O\left(\frac{1}{\omega}\right) = \frac{1}{\kappa} + O\left(\frac{1}{\omega}\right)$$

$$G_{\mu\nu} = -\frac{8\pi\kappa}{c^2} T_{\mu\nu} + O\left(\frac{1}{\omega}\right)$$

which explicitly illustrates the large ω limit.

11.6 Write the Brans-Dicke equation (11.138) as

$$G_{\mu\nu} = -\frac{8\pi}{c^2\varphi}(T_{\mu\nu} + B_{\mu\nu})$$

where $B_{\mu\nu}$ is interpreted as the energy-momentum tensor of the scalar field. Show that $B_{\mu\nu}$ is divergenceless from its definition.

Problems

11.1 The derivation of the geodesic equation of motion presented in the text concerned a structureless body, a small dust globule. Consider a small spinning body and obtain an equation of motion (see Papapetrou, 1951).

11.2 Consider an extended body and discuss corrections to the geodesic equation of motion. How might one study the motion of two bodies of comparable mass interacting gravitationally? What would be the effect of gravitational radiation on the motion?

11.3 The energy-momentum of a gravitating system was obtained as a surface integral in (11.102). A similar procedure can be used to obtain the angular momentum of a gravitating system (Cohen, 1968). Use this approach to identify the parameter a in the Kerr metric, as discussed in Sec. 7.7.

11.4 Obtain the energy momentum tensor of the gravitational field using canonical field theory (see Bjorken and Drell, 1965, for a discussion of the canonical theory in flat space).

11.5 Obtain the exterior field of a spherically symmetric body in Brans-Dicke theory, analogous to the Schwarzschild solution.

11.6 Obtain expressions for the planetary perihelion shift, the solar deflection of light, and time delay of radar pulses for the Brans-Dicke theory, in analogy with the results of Chap. 6 for the Schwarzschild solution. What of the gravitational red shift prediction? (See Weinberg, 1972.)

11.7 Compare the Brans-Dicke expressions from the above with the general relativistic expressions and with the observational values discussed in Chap. 6. What range of values may ω take for each test if the scalar tensor theory is to be consistent with observation?

Bibliography

Bergmann, P. G. (1942): "An Introduction to the Theory of Relativity," New York.
Brans, C., and R. H. Dicke (1961): Mach's Principle and a Relativistic Theory of Gravity, *Phys. Rev.*, **124**:925.
Bjorken, J. D., and S. D. Drell (1965): Relativistic Quantum Fields, New York.
Cohen, J. M. (1968): Angular Momentum and the Kerr Metric, *J. Math. Phys.*, **9**:905.
Einstein, A., and J. Grommer (1927): Allgemeine Relativitätstheorie und Bewegungsgesetz, *Sitzber. Preuss. Akad. Wiss. Berlin*, pp. 2–13.

Einstein, A., and L. Infeld (1940): The Gravitational Equations and the Problem of Motion, II, *Ann. Math.*, **41**(2):455–464.
Einstein, A., L. Infeld, and B. Hoffmann (1938): The Gravitational Equations and the Problem of Motion, *Ann. Math.*, **39**(2):pp. 65–100.
Fock, V. (1959): "The Theory of Space, Time and Gravitation," New York.
Infeld, L., and J. Plebanski (1960): "Motion and Relativity," London.
Levi-Civita, T. (1964): "The N-Body Problem in General Relativity," Dordrecht, chap. 1.
Palatini, A. (1919): Deduzioni invariantive delle equazioni gravitazionali dal principio di Hamilton, *Rend. Circ. Mat. Palermo*, **43**:203–212.
Papapetrou, A. (1951): Spinning Test Particles in General Relativity I, *Proc. Roy. Soc.*, **209**:248.
Trautman, A. (1962): Conservation Laws in General Relativity, in L. Witten (ed.) "Gravitation: An Introduction to Current Research," New York, pp. 169–198.
Weinberg, S. (1972): "Gravitation and Cosmology," New York, chap. 7.

See also the standard books on general relativity theory.

CHAPTER

12

Descriptive Cosmic Astronomy

In the various phenomena which physicists investigate a small number of fundamentally different types of interactions occur; we observe the effects of the "strong" and "weak" nuclear interactions, the electromagnetic interactions, and the gravitational interactions. Of these the gravitational interactions are by far the weakest. Nevertheless, because the nuclear interactions are of short range and large aggregates of matter tend to be electrically neutral, it appears that the large-scale phenomena of the universe are most strongly affected by the gravitational interaction. If gravity is indeed the dominating influence, then as a theory of gravity, general relativity should be able to give a description of the universe in the large. The cosmological problem within the framework of general relativity consists in finding a model of the universe as a whole which is a solution of Einstein's equations. Such a model is intended to describe *only* the large-scale state of the universe; for example, the energy-producing processes which take place inside stars are of nuclear origin and clearly cannot be described by relativity theory. Moreover, the process of galactic evolution, although largely determined by the gravitational interaction, is on a much smaller scale than we wish to consider and must be separated from the cosmological problem. An acceptable cosmological model need therefore fit only a limited number of observational facts of a global nature. We shall first discuss the observational facts that a cosmological model should describe, and in particular the presently determined values of some important cosmological parameters. In the rest of the chapter we shall discuss some general features of the cosmological problem, preparatory to discussing specific cosmological models in the next chapter.

12.1 Observational Background

In astronomical studies one characterizes a star or galaxy by position coordinates. These provide a marker system in the non-Euclidean space which we use for theoretical study. In particular, one attaches to a star or galaxy a radial distance marker referred to as its *astronomical distance*. For the nearest stars the astronomical distance may be determined by triangulation, using local Euclidean geometry; the base line is the size of the earth's orbit, and the angle is the star's parallax. This method is the oldest and simplest but is limited to a few thousand stars within about 100 light years, since more distant stars have too small a parallax to measure accurately. In particular it is certainly limited to stars in our own galaxy and cannot directly yield information of cosmological interest (see Prob. 12.1, however).

For more distant stars we must proceed differently. To do this we make use of the inverse-square law of decreasing light intensity. Suppose we know that a distant star (or galaxy) has the same total rate of emission, or *absolute luminosity*, as another closer star whose distance L_0 is known; calling E and E_0 the apparent luminosities of the two stars, we can define an astronomical distance L for the more distant star by the relation

$$\frac{L^2}{L_0^2} = \frac{E_0}{E} \tag{12.1}$$

It is known that the surface temperature (or equivalently the spectral type) and the absolute luminosity of the so-called *main-sequence* stars are correlated in a simple way. (This is usually shown on a plot of absolute luminosity versus surface temperature known as a *Hertzsprung-Russell diagram.*) This correlation allows one to use relation (12.1) in practice by comparing two stars of the same spectral type.

For larger distances certain variable stars may be used as markers; such stars possess an absolute luminosity that varies periodically in time, with a period that is directly related to their absolute luminosity. The most important of these are the classical Cepheid variables (Leavitt, 1912). Such stars can be seen in other galaxies. Indeed Cepheids led to the first confirmation of the extragalactic nature of spiral nebulae such as M31 in Andromeda, a galaxy about 2×10^6 light years away. The use of Cepheids as distance markers has been historically fraught with difficulty and confusion; e.g., there are two distinct types of Cepheids. It is hoped that the calibration problem has now been cleared up. This difficulty in the use of Cepheids is one of the reasons for the large changes in our estimates of the scale of the universe during the twentieth century.

For more distant galaxies, those outside the local cluster of galaxies in which we reside, we need brighter stars than the Cepheids. The distance markers used are the supergiant blue stars, or novae which flare up briefly to approximately known absolute magnitudes, or entire globular clusters of stars.

Finally, to obtain distance markers of cosmological significance one must use entire galaxies since no individual stars are sufficiently bright. For this purpose certain types of galaxies whose absolute luminosities appear to be well defined are used. One representative method is to note that galaxies tend to occur in clusters of hundreds or thousands; it is then reasonable to suppose that the brightest galaxies (generally the elliptical, or E, galaxies) in various clusters all have approximately the same absolute luminosity, which provides a reference luminosity. A more detailed discussion is given by Hubble (1957) and by Sandage (1961), and a briefer summary by Weinberg (1972).

In summary, the procedure used to define the astronomical distance L is to extend all over the world a radial *marker system L in which the law of energy decrease with the inverse square of the distance is strictly true* and which coincides with the radial coordinate used in our local Euclidean geometry at small distances. The distance marker L thus defined is sometimes called an "energy distance."

At the beginning of Chap. 2, we mentioned briefly the concept of a Riemannian manifold which is pieced together by overlapping coordinate neighborhoods with known transformation laws for transition from one mapping of the manifold into another. The less mathematically inclined reader might have considered our definition as an axiomatical-logical refinement of no great practical value in experimental science. It may therefore be illuminating to point out at this stage how close observational and measuring procedures are to this concept of overlapping coordinate sets. In the above discussion we mention first the neighborhood of the earth, where distances can be measured by parallaxes and angular readings. In this neighborhood we obtain the main-sequence correlation referred to above and extend the distance scale to where we are able to gauge the Cepheid variables and establish their luminosity-periodicity law. Then these variable stars can be used as markers far beyond the first coordinate map. In the new piece of the cosmos opened up for measurement, we discover the spectral-type laws which enable us by a further extension to formulate the Hubble distance–red shift relation discussed below, which opens up new coordinate possibilities. The mathematical concept of a pieced-together manifold is precisely a formulation of these classical astronomical practices.

While determining the apparent luminosity of stars, astronomers also determined and classified their line spectra and discovered that a shift of the known spectral lines toward the red occurred for stars in galaxies which were farther and farther away. For each distant galaxy, the relative shift in wavelength $\Delta\lambda/\lambda$ appeared to be proportional to the astronomical distance of the galaxy as defined above. We shall see later that this striking phenomenon can be successfully interpreted as a recession of distant galaxies whose speed increases linearly with their distance from us. One can then visualize the universe as *expanding*. Most of the original astronomical observations of this effect are due to Hubble (1936), and the results can be summarized by the simple relation

$$\frac{\Delta\lambda}{\lambda} \cong \frac{L}{c} H \tag{12.2}$$

where the approximate equality sign reflects experimental uncertainty. This relation is known as Hubble's law, and the important number H is called Hubble's constant. Its first evaluation in 1936 gave a value $H^{-1} = 0.56 \times 10^{17}$ s, which was used in the literature until 1958. This number, which may be interpreted as a fundamental distance or time scale of the universe, has undergone considerable change since then. Sandage pointed out in 1958 that Hubble had apparently confused interstellar hydrogen clouds with individual blue stars in his original estimate of H. Indeed, if this and all known sources of error and uncertainty are combined, the value of H should be revised downward by nearly a factor of 10 and assigned a rather large uncertainty. The value of H in present use is (Sandage, 1972)

$$H^{-1} = (5.6 \pm 0.6) \times 10^{17} \text{ s} = (1.8 \pm 0.2) \times 10^{10} \text{ years}$$

$$\frac{\Delta\lambda}{\lambda} = \frac{L'}{c} H \tag{12.3}$$

with a given fixed value of H is strictly true. This system of markers, called *Hubble distances*, can be extended to very remote galaxies for which an "energy distance" is very difficult to estimate accurately because of very low apparent luminosity and because of selective light absorption in intergalactic space. A Hubble distance, on the other hand, can be obtained very accurately because the shift in wavelength can be measured with high precision as soon as one spectral line is identified.

Very interesting objects, called *quasi-stellar objects* or *quasars*, many of which have extraordinary red shifts, have lately been observed. The greatest red shift reported so far is $\Delta\lambda/\lambda = 3.5$, which is so large that time-dilation effects are important and (12.3) should not be used to

define its distance (see Chap. 4). It is clear that if this red shift is actually caused by the cosmological expansion of the universe, the quasar must be far away indeed, about 95 per cent of the way to the "edge" of the universe, i.e., the distance at which the galaxies recede from us at the velocity of light and are unobservable. Such an object is obviously of great cosmological interest. Unfortunately the physical nature of the quasars is not yet understood; since they vary in luminosity in typical times of about a month they cannot be much larger than a light month in size, which is very much smaller than the 10,000–light year size of a typical galaxy. On the other hand their absolute luminosity appears to range from 1 to 100 times that of a typical galaxy. The question of how such a small object can radiate so much energy has presented theoreticians with very interesting and challenging problems which are not yet solved. The role of quasars in cosmology at the present time is uncertain, but their study may lead to drastic changes in our concepts of the evolution of galaxies, the early universe, and cosmology (see Morrison, 1973).

To apply Einstein's equations to the actual universe with a precise physical distribution of matter represented by an exact energy-momentum tensor $T^{\mu\nu}$ would obviously be a hopeless mathematical task. In order to achieve a tractable description of the matter distribution in the universe in the large, theoretical cosmologists always make the idealizing assumption that, on a sufficiently large scale, matter can be considered to be homogeneously distributed. Observationally, out to the largest distances reached by present-day telescopes, this uniformity appears on the scale of the clusters of galaxies, but not on the scale of individual galaxies whose clustering tendencies are quite evident (Oort, 1958). One may then consider an idealized universe of uniform continuous matter distribution represented by a constant density of matter energy ρ_0 which serves as the $T^0{}_0$ component of the energy-momentum tensor.

Estimates of the average density of the universe are very difficult. To obtain the density due to the ponderable matter in galaxies one must count galaxies out to some Hubble distance, divide by the volume in which they are contained, and multiply by their average mass. The determination of the volume requires a knowledge of Hubble's constant, which is not known with great accuracy. To obtain the masses of galaxies one may, for example, analyze the relative velocities and separations of a pair of galaxies, a process which is subject to considerable uncertainty. Using such methods it is estimated (Oort, 1958) that $\rho_0 = 2 \times 10^{-31}$ g/cm³. This number could easily be in error by a factor of 3. In addition to the ponderable matter in galaxies ρ_0 should contain the density of intergalactic dust and gas (Beer, 1960), dim intergalactic stars, black holes, particles such as cosmic rays, neutrinos, the quanta of gravitational radiation (called *gravitons*), photons, and possibly others. It is difficult to estimate the density of such material or even to get a reasonable upper limit. For example, the only upper limit on the neutrino density that would result from inverse β-decay reactions in stars is several orders of magnitude greater than the above estimate for the galactic ponderable mass (Pontecorvo and Smorodinsky, 1962). As a result it is impossible at present to place a safe upper limit on ρ_0; it could easily be as large as 10^{-28} g/cm^3. As we shall discuss later in Sec. 13.3, there is observational evidence which, when combined with theory, suggests that the mass density should be about 0.5×10^{-29} g/cm^3 or greater.

The field equations for a space filled homogeneously with matter lead to an evolution of the universe in time, and this evolution can be compared with astronomical observations. Some models correspond to a static universe (all of which are unstable, however) and some to a dynamic universe with limited life span. In the latter case, there is an origin of time, and the time elapsed to attain the present state of the universe should be compatible with experimental estimates of "ages," or periods of evolution, of different parts of the universe. The age of the universe should, for instance, be larger than the age of the earth and the solar system, which can be determined from the ratios of abundances of radioactive substance to decay products in various rocks and meteorites. This method leads to an estimate of the earth's age as $(4.55 \pm 0.07) \times 10^9$ years (Patterson, 1960). The age of the universe should also be greater than or equal to the age of the oldest stars. When plotting the usual color-absolute magnitude diagram for stars in our galaxy, one finds some stars that are evidently 10^{10} years old (Wilson, 1950); for other samples of stars, this same age and ages up to 3.2×10^{10} years have been found (Oke, 1959).

The fact that relativity predicts an evolutionary universe was actually recognized by Einstein but rejected as physically unreasonable. He then introduced a "cosmological constant" to obtain a static universe (Sec. 13.2). Only later was it discovered that the universe does indeed seem to be expanding and evolving, thus making the ad hoc introduction of the cosmological constant unnecessary. Further evidence for the evolutionary nature of the universe was obtained in 1965 by Penzias and Wilson (1965). They discovered that space appears to be filled with electromagnetic radiation with a blackbody spectrum, implying that it is in thermal equilibrium, with a temperature at present of about 3°K. Dicke and collaborators (1965) suggested that this radiation is actually the cooled remnant of a primordial fireball that accompanied the explosive birth of the universe about 2×10^{10} years ago, a time equal

to the inverse of Hubble's constant. The consequences of such a "big bang" birth of the universe had been investigated by Gamow, Alpher, and others many years before (Gamow, 1946; Alpher, 1948). Further measurements of the radiation intensity confirmed the character of the blackbody spectrum, as summarized by Dicke (1970), thus lending strong support to the validity of an expanding-universe concept with a big-bang birth. Conversely it is evidence against the steady-state model, which we discuss in Sec. 13.5.

12.2 The Mathematical Problem in Outline

The mathematical task of solving the cosmological problem consists in determining a large-scale metric of the four-dimensional world and a corresponding large-scale mass-energy distribution satisfying Einstein's equations. The metric, in turn, gives physical predictions since the galaxies and light rays move along geodesics of the four-dimensional space which it defines. In the present case, one has to solve a problem in the large in contrast to the local Schwarzschild problem treated earlier; in the Schwarzschild case, a metric was determined from a priori knowledge of the energy-momentum tensor, which was zero except at one single point; in the present case the metric and the energy density have to be determined together over the four-dimensional world. When the cosmological problem is solved, it will give the large-scale average behavior of the physical world, and therefore give the boundary condition to be used at infinity for local models like the Schwarzschild solution.

To restrict the possible forms of a cosmological metric, we shall first impose the requirement of spatial isotropy, which we have seen appears to be physically justifiable on a large enough scale. Then different models can be developed corresponding to either static or time-dependent solutions and to different values of a characteristic parameter which will appear in the general form of the isotropic metric and is related to the curvature of the three-dimensional world. From each such model one can derive properties of the corresponding physical world. A comparison of these predictions with our observational knowledge should provide a test of the validity of the model. Unfortunately, the general imprecision of the experimental astronomical data relevant to cosmological problems tends to make one consider any attempt at comparison with theoretical models easy or difficult, according to one's individual scientific integrity.

We have mentioned, in Chap. 10, that the most general form of Einstein's equations under certain reasonable mathematical requirements is

$$R^{\nu}{}_{\mu} - \tfrac{1}{2}Rg^{\nu}{}_{\mu} + \Lambda g^{\nu}{}_{\mu} = -\frac{8\pi\kappa}{c^2}T^{\nu}{}_{\mu} \qquad (12.4)$$

where Λ must be a constant, the so-called cosmological constant. We solved these equations earlier (Chap. 6), neglecting the cosmological term when studying the Schwarzschild solution. This solution was seen to be in satisfactory agreement with the available observational tests within the solar system. In Chaps. 10 and 11 we also made investigations of these equations, assuming Λ to be zero. We shall now discuss the complete equations.

We notice at once that the general form of Einstein's equations (12.4) with $\Lambda \neq 0$ does not admit a flat-space solution for an empty universe. This is evident since an empty universe by definition is characterized by $T^{\nu}{}_{\mu} = 0$ and a flat space is characterized by $R^{\nu}{}_{\mu} = 0$; therefore Eqs. (12.4) cannot be satisfied unless $\Lambda = 0$. That is, an empty space cannot be described mathematically using Euclidean geometry unless $\Lambda = 0$.

In our previous treatment of problems in gravitational theory, however, we considered masses isolated in a finite part of the universe and tried to solve the field equations with the boundary condition of Euclidean geometry at infinity. This was based on the assumption that the influence of the finite-mass distribution is negligible at infinity and that in empty space Euclidean geometry prevails. In other words, we assumed then that an empty space is flat. Our theoretical predictions for phenomena within the solar system were in agreement with experiment. Therefore, on the scale of the solar system, the presence of a Λ-term in the general form of Einstein's equations must have a negligible influence; thus, if nonzero, Λ must at least be very small.

This can also be expected from dimensionality considerations. In any system of coordinates x^{μ} which have the dimensions of length, the $g_{\mu\nu}$ are dimensionless from the definition (1.2) of a line element. The elements of the contracted Riemann tensor then have the dimension of inverse length squared since they are obtained from the $g_{\mu\nu}$'s by double differentiation. To preserve homogeneity in (12.4) it is therefore necessary for Λ to have the dimension of the square of an inverse length. Indeed, we shall see presently that the integration of (12.4) for an empty space leads to a finite universe with a "characteristic size" of $1/\sqrt{\Lambda}$. On dimensional grounds one might expect from this that Λ should in general be related to the inverse square of the size of the universe and should therefore play a negligible role in local problems.

Let us digress for a moment to show that this statement can be illustrated in a simple way by considering the Schwarzschild problem with the inclusion of the cosmological term $\Lambda g_{\mu\nu}$. Setting $T_{\mu}{}^{\nu} = 0$ in (12.4), we find by contraction that $R = 4\Lambda$, and hence that

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{12.5}$$

Let us briefly indicate how to solve this equation in the case of a static radially symmetric metric as given in (6.9) and (6.26). We observe that the expression for the contracted Riemann tensor $R_{\mu\nu}$ is the same as in our calculations of Chap. 6. We conclude from (6.31) and the equation $R_{00} = \Lambda g_{00}$ that

$$\nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\nu'\lambda' + \frac{2\nu'}{r} = -2\Lambda e^{\lambda} \tag{12.6}$$

and from (6.35) and $R_{11} = \Lambda g_{11}$,

$$\nu'' + \tfrac{1}{2}\nu'^2 - \tfrac{1}{2}\nu'\lambda' - \frac{2\lambda'}{r} = -2\Lambda e^{\lambda} \tag{12.7}$$

Thus subtraction of (12.7) from (12.6) yields, as in the original Schwarzschild problem,

$$\nu' + \lambda' = 0 \qquad \nu + \lambda = \text{const} = \log k \tag{12.8}$$

We have, however, no possibility of determining the value of this constant since we cannot ask for a Lorentzian character of the solution at infinity. We write, therefore,

$$\lambda = \log k - \nu \tag{12.9}$$

and insert into (12.6) to obtain

$$(e^{\nu})'' + \frac{2}{r}(e^{\nu})' = \frac{1}{r}(re^{\nu})'' = -2\Lambda k \tag{12.10}$$

This yields the integral

$$re^{\nu} = \alpha + \beta r - \Lambda\frac{kr^3}{3} \tag{12.11}$$

with α, β as constants of integration.

Next we use (6.44) and the field equation $R_{22} = \Lambda g_{22}$ to obtain

$$\frac{\partial^2}{\partial\theta^2}(\log|\sin\theta|) + (e^{-\lambda}r)' - 2e^{-\lambda} + \cot^2\theta + e^{-\lambda}r\left(\frac{\lambda' + \nu'}{2} + \frac{2}{r}\right) = -\Lambda r^2 \tag{12.12}$$

Rearranging and using (12.8), we reduce this to

$$(e^{-\lambda}r)' = 1 - \Lambda r^2 \tag{12.13}$$

which, by virtue of (12.9), is the same as

$$(re^{\nu})' = k(1 - \Lambda r^2) \tag{12.14}$$

Comparing (12.11) and (12.14), we conclude that

$$k = \beta \tag{12.15}$$

Thus, from (12.9) and (12.11), it follows that

$$e^{\nu} = k\left(1 + \frac{\alpha}{kr} - \Lambda\frac{r^2}{3}\right) \qquad e^{-\lambda} = \left(1 + \frac{\alpha}{kr} - \Lambda\frac{r^2}{3}\right) \tag{12.16}$$

In view of the form of the line-element (6.9) we can always choose our unit of time such that $k = 1$, and therefore

$$e^{\nu} = 1 + \frac{\alpha}{r} - \Lambda\frac{r^2}{3} = e^{-\lambda} \tag{12.17}$$

Thus our resulting solution is identical with (6.53) except that the term $(1 - 2m/r)$ has to be replaced by $(1 - 2m/r - \frac{1}{3}\Lambda r^2)$.

We remark, first, that in this solution we do not obtain the ordinary Lorentzian metric if we suppose it to be regular everywhere, i.e., if we assume $m = 0$. In order to interpret the geometry of the line element which belongs to a radially symmetric static solution that is regular at the center (that is, with $m = 0$), let us consider the spatial part of the line element

$$dl^2 = \frac{dr^2}{1 - \frac{1}{3}\Lambda r^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \tag{12.18}$$

As we shall see in the next section [formula (12.62)], this line element corresponds to the metric on the three-dimensional surface of a four-dimensional hypersphere with radius $R = \sqrt{3/\Lambda}$. This shows clearly the relation between Λ and the size R of the finite universe described by (12.18). Obviously, for r much smaller than the size of the universe, the presence of the extra term $\frac{1}{3}\Lambda r^2$ in the Schwarzschild solution may be neglected. Thus observation within our planetary system would not enable us to determine the presence or absence of the Λ-term in the field equations.

We shall later discuss various models of isotropic worlds which are empty or filled homogeneously with matter. It should be pointed out here that the present Schwarzschild solution with $m = 0$ does not describe an empty isotropic model of this kind. While we can easily attain spatial isotropy by a proper change of the radial variable, the center of radial symmetry always plays a distinguished role in the time term of the line element. Furthermore, there occurs a radial force directed away from this center of symmetry, which we can easily demonstrate explicitly. For small values of m we may use the perturbation theory of Sec. 4.3 to interpret the dynamical consequences of the Schwarzschild metric in the case of a nonvanishing cosmological term Λ. We have, on the one hand,

$$g_{00} = 1 - \frac{2m}{r} - \tfrac{1}{3}\Lambda r^2 \tag{12.19}$$

and on the other, by (4.142),

$$g_{00} = 1 + \frac{2\varphi}{c^2} \tag{12.20}$$

where $\varphi(x^i)$ is the potential in classical mechanics which would induce motion of a test particle approximately along the geodesics of the metric considered. Thus we obtain the approximate correspondence

$$\varphi = -\frac{\kappa M}{r} - \tfrac{1}{6}\Lambda c^2 r^2 \tag{12.21}$$

Therefore, even in the absence of the mass M at the origin, a test particle would be subjected to a radial acceleration

$$a = \tfrac{1}{3}\Lambda c^2 r \tag{12.22}$$

This shows that the origin of the coordinate system is dynamically distinguished from all other points in space, and is not an arbitrary point in three-dimensional space.

One last remark about the solution (12.17) and the space line element (12.18) for $m = 0$ can be made. The solution could be realized by some sort of radially symmetric distribution of matter about the origin; the neighborhood of the origin where the solution holds must of course be free of matter. Thus one arrives at a notion of a spherical cavity inside some spherical distribution of mass as having a metric solution (12.17). It is thus easily seen that, for $\Lambda = 0$, one obtains a pseudo-Euclidean metric, while for $\Lambda \neq 0$, we obtain a non-Euclidean geometry (12.18). Thus in principle the acceleration (12.22) inside a mass shell could serve to measure the cosmological constant Λ.

From the foregoing example we see that the original requirement of flat space at infinity need not be retained when dealing with cosmological problems; we are still ignorant of the large-scale topology of physical three-dimensional space and of the four-dimensional space-time manifold. Thus it is clear that one cannot definitely decide whether or not the cosmological term $\Lambda g_{\mu\nu}$ in Einstein's equations should be set equal to zero. We shall present models for the two cases separately. However, in both cases we shall restrict from the beginning the functional form of the potentials $g_{\mu\nu}$ by certain postulates of symmetry which we shall discuss next.

12.3 The Robertson-Walker Metric

We mentioned earlier that the large-scale distribution of extragalactic nebular clusters in space is roughly isotropic around our own galaxy and that the number of clusters in a given volume seems to remain constant everywhere: the larger the scale considered, the better the approximation. To simplify the mathematical description, we therefore make the physical assumption that matter is distributed homogeneously in the world. Furthermore, we desire that the geometry of space be determined by the matter distribution. This general requirement is known as Mach's principle. It is the name given to it by Einstein (1918) when generalizing Mach's original hypothesis that the inertia of one body is due to the presence of all other bodies in the universe (Mach, 1883). Einstein's equations are one possible particularization of this principle: $T_{\mu\nu}$ characterizes the matter-energy distribution, and $G_{\mu\nu}$ the space geometry.

On the basis of Mach's principle, we shall require that the geometry of the three-dimensional space be homogeneous like the matter distribution. We shall thus start with a purely geometrical study of the form of a metric which describes a four-dimensional space containing a three-dimensional subspace of homogeneous geometry. The precise link with the physical assumption of homogeneity of the mass distribution will be elucidated afterwards through a specialization of the coordinate system used.

The following mathematical hypotheses are our starting point:

1. There exists a global time-coordinate which serves as the x^0 of a Gaussian coordinate system as defined in Chap. 2. [See Eq. (2.28).]
2. The three-dimensional spaces belonging to various constant values of this time-coordinate are locally isotropic.
3. Any two points in a three-space belonging to a given fixed time are equivalent.

Let us first remark that the use of a distinguished time-coordinate marks the abandonment of a completely covariant treatment of the cosmological problem. This is the price one has to pay to simplify the cosmological models and to describe physical reality in convenient mathematical terms. The fact that x^0 is a Gaussian time-coordinate means that, at some given moment, the matter of the universe is on the average at rest in its distinguished three-space. It therefore describes geodesic trajectories which are orthogonal to that three-space. As was shown in Chap. 2, these geodesics will remain parallel to the x^0 axis in all future three-spaces. That is, at any moment the matter of the universe will be at rest in this distinguished three-space. We may therefore refer to this coordinate system as the *co-moving system*. This point will be discussed in more detail in the next section.

The second condition is a local requirement, which is all that is necessary in most mathematical problems of general relativity theory. Let us consider an *arbitrary* but fixed point in space-time. As we discussed in Sec. 6.2, local isotropy implies that the space coordinates must appear in the line element ds^2 in the spherically symmetric combination

$$d\sigma^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{12.23}$$

or equivalently, in r, θ, φ coordinates centered about the given point

$$d\sigma^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \tag{12.24}$$

The complete line element in our Gaussian coordinates is therefore expressible as

$$ds^2 = (dx^0)^2 - e^{G(x^0,r)}\, d\sigma^2 \tag{12.25}$$

where the function G cannot depend on θ and φ because of the assumed isotropy of three-space around the given point. Writing ds^2 in the form (12.25), with an exponential function, enables us to keep the signature in evidence and guarantees that the determinant of $g_{\mu\nu}$ does not vanish.

Finally, the third postulate, the equivalence of all points in three-space at all times, requires that two observers at two different points observe a similar physics; the only thing which may differ between the two observers is the measuring scale they use. Therefore the ratio of physical or proper-distance elements at two points in space defined, respectively, by the coordinates (r_1,θ_1,φ_1) and (r_2,θ_2,φ_2) must remain fixed